Moduli space of supersymmetric black holes in five dimensions

Veronika Breunhölder

Doctor of Philosophy
University of Edinburgh
2019

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text. No part of this thesis has been submitted here or elsewhere for any other degree or qualification. Chapters 2–4 of this thesis are based on the published papers [16, 17] authored by myself and my supervisor James Lucietti.

(*Veronika Breunhölder*)

Acknowledgements

First of all, I want to thank my supervisor, James Lucietti, for his continued support and guidance throughout my PhD, in particular for suggesting the topic of this thesis, for many meetings and answered questions, and for his help during the final stages of completing this work. Not least I would also like to thank him for the many enjoyable conversations in between technical discussions.

I also want to thank the Mathematical Physics Group here in Edinburgh, specifically Joan Simón and José Figueroa-O'Farrill, for many interesting seminars, working groups, or lunch meetings, and for the warm welcome when arriving here.

I further want to express my gratitude to the EPSRC for awarding me a studentship to fund this PhD.

Any period in life is shaped by the people one is surrounded with, and I am forever grateful to those who have shaped this one. In particular I want to thank Carlos, Dany, Jenny, Lukas, Matt, Ruth, Simon, Tim, and Xīlíng, for bread and cake, gigantic snowmen, ballet, a half-marathon, many Friday pub nights, music—and in general for the bittersweetness in completing this thesis.

Last but not least, I want to thank my family, for their support, and Alex, for continuing to bring out the best in me.

Lay Summary

The existence of black holes was certainly one of the most intriguing predictions of Einstein's theory of general relativity, and it is fair to say that the fascination surrounding them has not declined. Not only are black holes still a major area of research in both theoretical and experimental physics, but they have also, within the four years of this PhD, in fact twice made it onto the front cover of public newspapers all across the world: First, in early 2016, black holes were in the spotlight for the first ever direct observation of gravitational waves—ripples in spacetime produced in the final stage of a merger of two black holes—by the Laser Interferometer Gravitational-wave Observatory (LIGO) [1]. The second occasion that gained a lot of attention for black hole research, was the very recently published, first ever "picture" taken of a black hole by the Event Horizon Telescope (EHT), a collection of radio telescopes spread across the globe [2] (see Figure 1).

What is so special about black holes? Black holes are objects whose gravitational force is so strong that not even light can escape from them. Thus looking at a black hole from a distance, it will appear as a black shadow in the sky, giving it its name. The edge of the black hole, which is the boundary of the region from which nothing can escape, is called the black hole's *event horizon*. It was shown by Stephen Hawking in the 1970s that this event horizon must always have the shape of a sphere. This does not have to be a round sphere, it might be squashed, but there can never be any holes in it (like for example in a doughnut shape).

Black holes have some curious features. One of the most famous of these is what is often called the *no hair theorem*. The no hair theorem says that a black hole only depends on three parameters: its mass, its electric charge (if it has one), and its angular momentum (if it is spinning). There are no other characteristics (these would be the "hair"). What this means is that for someone looking at two black holes which have the same values for those three parameters, there is no way of telling them apart: they look exactly the same. This certainly gives us quite a good understanding of the types of black holes there are, and how they could look like: all we need to know is their mass, charge, and angular momentum. Thus, black holes are completely *classified*; we can write down a list of all possible types of black holes there are.

One of the main insights of general relativity was that the universe we live in should be viewed as what is called a four-dimensional *spacetime*. Three of these four

Figure 1: The image of the M87* black hole captured by the Event Horizon Telescope. Photograph: EHT Collaboration.

dimensions are the spatial dimensions we see around us all the time (left/right, forwards/backwards, and up/down). The fourth is the time dimension. Other than with the spatial dimensions, we cannot choose to stand still or even move backward in time, but rather we are constantly moving along that direction into the future. Now all that was discussed above are rules for black holes sitting in such a four-dimensional universe. What would happen if a black hole was instead sitting in a higher-dimensional spacetime, for example one where in addition to left/right, forwards/backwards, and up/down, there is another direction of space that one can move in? This sounds like a very theoretical question to ask, but for example string theory, which many believe could be the ultimate theory describing our universe, actually says that there *must* be such extra dimensions—they are just curled up and too small for us to see.

Simply put, the answer to the above question is that things get a lot more complicated when we want to study black holes in higher (that is, more than four) dimensions. First of all, it was found that higher-dimensional black holes no longer have to be spherical. For example, one can have ring-shaped black objects (so called *black rings*), or even combinations of black holes of different shapes, like a spherical black hole sitting inside a black ring (called a *black saturn*). Further, the no hair theorem breaks down in higher dimensions. That means we can no longer specify black holes by just a simple set of parameters. In fact we no longer have a classification of solutions at all—we do not even know what types of black holes exactly exist!

This thesis tries to address this problem for the special case of five-dimensional black holes. While a full classification of all black holes in five dimensions is out of reach, with some assumptions we succeed in giving a classification of a special class of black holes: so called *supersymmetric* black holes.

Abstract

This thesis presents a classification of all asymptotically flat, supersymmetric and bi-axisymmetric (i.e. possessing a $U(1)^2$-symmetry) soliton and black hole solutions to five-dimensional minimal supergravity. In particular, by combining local constraints from supersymmetry of the solutions with global constraints for stationary and bi-axisymmetric spacetimes, we show that any solution must be multi-centred with a Gibbons–Hawking base. We also find a refinement of the allowed horizon topologies of this class of solutions, to one of S^3, $S^1 \times S^2$ or a lens space $L(p,1)$. We construct the general, smooth solution associated with each possible rod structure, thereby finding a large moduli space of black hole spacetimes with noncontractible 2-cycles in the domain of outer communication. This includes examples for each of the allowed horizon topologies. In the absence of a black hole we obtain a classification of the known "bubbling" soliton spacetimes.

We then move on to a systematic analysis of the subclass of three-centred solutions contained in the constructed moduli space, focusing on the special case of single black hole solutions. This class is composed of seven regular black hole solutions. We find that four of these can have the same conserved charges as the original spherical, supersymmetric black hole, the BMPV solution. These consist of a black lens with $L(3,1)$ horizon topology and three distinct families of spherical black holes with nontrivial topology outside the horizon. The former provides the first example of a nonspherical black hole with the same conserved charges as the BMPV black hole. Moreover, of these four solutions, three can have a greater entropy than the BMPV black hole near the BMPV upper spin bound. One of these is a previously known spherical black hole with nontrivial topology; the other two are new examples of a spherical black hole with nontrivial topology and an $L(3,1)$ black lens.

Contents

Chapter 1

Introduction

Throughout the past fifty years, black holes (Figure 1.1) have continued to shape theoretical physics. One of the reasons for them to be met with such great interest in the field, is that they play a special role in the quest for a unified theory of quantum gravity, see e.g. [62] for a review on the topic. This has its origins in the 1970s, when Bekenstein [8, 9] noted a striking similarity between the laws of thermodynamics and those describing the dynamics of a black hole, suggesting that black holes have thermodynamical properties like temperature or entropy. Around the same time Hawking proposed that, due to quantum effects near the horizon, black holes *evaporate*, emitting radiation consistent with a black body spectrum of temperature T_H proportional to the black hole's surface gravity [65]. This temperature was indeed in agreement with Bekenstein's findings. These observations sparked much debate over what is called the *black hole information paradox*: As the emitted radiation is thermal, it does not carry any information but the black hole's temperature. Thus, if a black hole evaporates entirely, any information about the states it was formed from is forever lost [66].

A definite resolution to that paradox can only be given by a complete theory of quantum gravity. String theory is a promising candidate for such a theory. It was naturally of interest to study black holes in the context of string theory. This was made possible in particular through the development of the theory of D-branes by

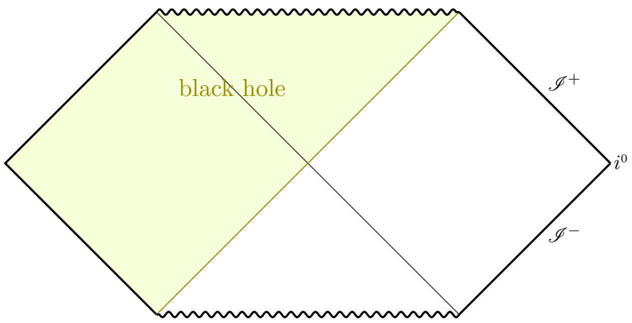


Figure 1.1: Definition of a black hole as $\mathcal{M} \setminus (\mathcal{M} \cap J^-(\mathscr{I}^+))$ where $\mathcal{M}$ is the physical spacetime manifold and $J^-(\mathscr{I}^+)$ denotes the causal past of future null infinity $\mathscr{I}^+$ (defined in the conformally compactified spacetime), see e.g. [101].

Polchinski in the 1990s [98]. One of the biggest successes of that period, and of string theory in general, was the derivation of the black hole entropy from string theory microstates [107].

Another breakthrough was marked by the discovery of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [90]. First conjectured as a duality between type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super-Yang–Mills theory on the four-dimensional boundary of AdS_5, this original example is now widely believed to be just one manifestation of an underlying fundamental principle, the holographic principle [78, 108]. The holographic principle proposes that any D-dimensional theory of quantum gravity has a dual quantum field theory in $D-1$ dimensions, living on the boundary of the D-dimensional spacetime. Black holes again play a special role, as they correspond to thermal equilibrium states in the boundary CFT. Much recent work in the field of AdS/CFT has been devoted to understanding (black hole) entropy from quantum information theory (*holographic entanglement entropy*, see [81, 102]).

The AdS/CFT conjecture was not the first time a connection was found between Anti-de Sitter space and conformal symmetries. In what is now often thought of as a precursor to AdS/CFT, in the 1980s Brown and Henneaux found that the asymptotic symmetry algebra of AdS_3—that is, the algebra generating the symmetries that preserve the asymptotic structure of the spacetime—is precisely given by two copies of the Virasoro algebra, the symmetry algebra of a two-dimensional CFT [18]. Asymptotic symmetries were also at the core of a recent holographic argument which claims to resolve the aforementioned information paradox [68]. It is argued therein that in fact black holes do have *soft hair*, that is, infinitely many conserved charges related to the BMS symmetry (the asymptotic symmetry group of Minkowski space [14, 103, 104]) at the boundary of the spacetime.[1]

Both in string theory, as well as in AdS/CFT and its applications, one is in general dealing with higher-dimensional theories of gravity, and thus in particular also with higher-dimensional black hole solutions. In addition, higher-dimensional black holes have in many ways proven to be interesting objects to study by themselves. The following sections give a review of some of the features of these solutions.

1.1 Preliminaries I: Black holes in higher dimensions

Higher-dimensional black hole solutions have shown to be much richer than their cousins in four or lower dimensions. This becomes manifest most notably in the lack of two important results which restrict possible solutions in four spacetime dimensions. First, Hawking's horizon topology theorem [64, 67] states that for black hole spacetimes in four dimensions cross-sections of the horizon must be of spherical topology. This fails

[1] The term "soft" comes from the fact that these charges, when acting on the vacuum, generate zero-energy—or soft—gravitons. The conservation laws associated with the charges are equivalent to the soft graviton theorem [119] in quantum field theory.

to be true in higher dimensions, as was first demonstrated by the construction of a five-dimensional black ring solution [41] with horizon topology $S^1 \times S^2$. Second, by the black hole uniqueness theorem (often dubbed "no-hair" theorem), in four spacetime dimensions a stationary, asymptotically flat, analytic black hole solution is completely determined by its mass, angular momentum and electric charge.[2] Once again, such uniqueness no longer exists once one enters the realm of more than four dimensions. Quite on the contrary, explicit counterexamples to such a result are known. As for the horizon topology theorem, the first such counterexample was provided by the black ring solution, which was found can carry the same asymptotic charges as the (topologically spherical) Myers–Perry black hole [94], the higher dimensional analogue of the Kerr solution.

Black hole non-uniqueness even occurs within the more restricted class of supersymmetric black holes. The first such example (with a connected horizon) was provided by the supersymmetric black ring [37, 38], which is not uniquely labelled by its conserved charges. However, the solution does not allow for conserved charges equal to that of the "standard" spherical, supersymmetric solution, the BMPV black hole [15]. A further counterexample, and the first example of a single black hole solution that does overlap with the spherical BMPV black hole was added by the construction of a supersymmetric spherical black hole with nontrivial spacetime topology in the domain of outer communication [85]. All these results do not, however, rule out the existence of uniqueness theorems that take into account additional internal degrees of freedom (including, but not necessarily limited to, information about horizon and spacetime topology). In fact, partial uniqueness theorems of that kind are known. In five spacetime dimensions, for instance, with the additional assumption of two axial Killing fields, a uniqueness theorem taking into account data about the fixed points of the symmetries (encoded in what is called the rod structure of the solution) has been proved [76]. Notwithstanding such successes, a full classification of asymptotically flat black hole solutions remains unknown for any dimension greater than four (see e.g. [73] for a detailed review on the topic).

The purpose of this section is to give a brief review on higher-dimensional black holes. The section is divided as follows. The first part, section 1.1.1, will discuss the topic of black hole uniqueness, both in four, as well as in higher dimensions. This also includes a discussion of Hawking's horizon topology theorem and its generalisation to higher dimensions. Section 1.1.2 will then give an overview of two known higher-dimensional black hole solutions, where we shall focus on non-supersymmetric solutions only. This includes the Myers–Perry black hole, as well as the black ring solution; supersymmetric solutions will be discussed at a later part in this chapter. Finally, 1.1.3 very briefly discusses thermodynamical properties of black holes.

[2] A detailed summary of the various uniqueness results in four dimensions can be found e.g. in [26, 69].

Note. Unless stated otherwise, in the following we assume black hole solutions to be asymptotically flat (in the Minkowskian sense).

1.1.1 Black hole uniqueness theorems

Horizon topology

Hawking's horizon topology theorem [64, 67] asserts that for a stationary, asymptotically flat black hole solution in four dimensions which satisfies the dominant energy condition, connected components of cross sections of the horizon must have topology S^2. This highly constrains the space of allowed solutions, and is a crucial ingredient in proving black hole uniqueness theorems in four dimensions. As such, it has naturally been of interest to try to generalise this result to higher dimensional spacetimes. One of the main obstacles presenting itself in that regard, is that Hawking's proof of the theorem relies on the dimension of the spacetime in the following crucial way. Let Σ be a spacelike hypersurface (equal-time slice) in a D-dimensional spacetime. A cross-section of the horizon then corresponds to a $(D-2)$-dimensional, closed, orientable hypersurface $\mathcal{H}$ in Σ, separating an "inside" from an "outside" region. The original argument goes along the following lines. Considering deformations of $\mathcal{H}$ along orthogonal directions and using the property that $\mathcal{H}$ constitutes a horizon, one finds a contradiction unless the total scalar curvature of $\mathcal{H}$ is non-negative (and in fact positive except for specific circumstances, cf. [64] for details on the strictly 4-dimensional case or [47] for general D). Now in four spacetime dimensions, $\dim \mathcal{H} = 2$, and one can apply the Gauss–Bonnet theorem to find that if the total scalar curvature of $\mathcal{H}$ is positive, so is its Euler-characteristic, $\chi(\mathcal{H}) > 0$. As $\mathcal{H}$ is closed and orientable, this means $\chi(\mathcal{H}) = 2$, i.e. $\mathcal{H}$ is homeomorphic to the two-sphere S^2, establishing the result for $D = 4$.

In more than four spacetime dimensions this argument fails, as Gauss–Bonnet can no longer be applied, and positivity of the total scalar curvature has no immediate topological consequences. This was however circumvented by Galloway and Schoen, who were able to prove the following generalised horizon topology theorem [47].[3]

Theorem 1.1. *Let $(\mathcal{M}, g)$ be a stationary spacetime of dimension $D \geq 4$ satisfying the dominant energy condition, and Σ, $\mathcal{H}$ as above. Then $\mathcal{H}$ is of positive Yamabe type.*

Positivity of the Yamabe invariant is equivalent to the manifold admitting a metric of everywhere positive scalar curvature. In $D = 4$ dimensions and with a closed and orientable $\mathcal{H}$, this is only the case for manifolds of topology S^2, consistent with the original horizon topology theorem. In the special case of five dimensions, which will be the main concern of this thesis, positivity of the Yamabe invariant is not as restrictive. Making use of the rigidity theorem [74], which implies that all analytic, stationary and rotating black hole solutions in five dimensions have at least one axial $U(1)$-Killing

[3] In fact [47] gives a more general version of this theorem, not assuming stationarity, however at the expense of some special cases having to be taken into account.

field, possible horizon topologies are found to be either S^3, $S^1 \times S^2$, lens spaces $L(p,q)$[4] as well as connected sums of these, or quotients of S^3 by certain finite subgroups of $SO(4)$ [72]. Assuming an additional axial Killing field, i.e. considering solutions with $\mathbb{R} \times U(1)^2$-symmetry, further reduces allowed horizon topologies to S^3, $S^1 \times S^2$ or $L(p,q)$ only [76]. All five-dimensional black hole solutions constructed so far possess such an $\mathbb{R} \times U(1)^2$-symmetry, and solutions with all three possible horizon topologies are known. Examples include the spherical Myers–Perry [94] (or in the supersymmetric case, BMPV [15]) black holes, the black ring solution and its generalisations [36, 37, 41, 99] with horizon topology $S^1 \times S^2$, or solutions with lens space topology which have been constructed and studied in [86, 88, 112].

Uniqueness in four dimensions: the Kerr–Newman family

Certainly one of the most striking results about black holes in four spacetime dimensions is their previously mentioned uniqueness, that is, the fact that in the stationary and asymptotically flat case they can be described entirely by a small set of physical parameters: their mass, electric charge and angular momentum. The full result, including rotating solutions, was first proved in the early 1980s by Mazur [91, 92] and independently by Bunting [19]. The result had been preceded by uniqueness theorems for static black hole solutions in Einstein(–Maxwell) theory (first established in [83, 84]), whose statements are given the following theorem.

Theorem 1.2. *Let $(\mathcal{M}, g)$ be an asymptotically flat, static (electro-)vacuum black hole solution with a non-degenerate event horizon. Then $(\mathcal{M}, g)$ is isometric to the Schwarzschild (Reissner–Nordström) solution, with $|Q| < M$ in the Einstein–Maxwell case.*

The vacuum case can be seen as follows (see e.g. [69] for a complete proof). Any static solution to the vacuum Einstein equations,

$$R_{\mu\nu} = 0\,, \tag{1.1}$$

can be written in the form

$$\mathrm{d}s^2 = -V^2\,\mathrm{d}t^2 + g^{(3)}\,, \tag{1.2}$$

where $g^{(3)}$ is a metric on a three-dimensional Riemannian manifold Σ, V is harmonic on Σ as follows from the 00-component of (1.1), and the three-dimensional Ricci-tensor satisfies the equations

$$R^{(3)}_{ij} = V^{-1}\nabla^{(3)}_i\nabla^{(3)}_j V\,. \tag{1.3}$$

This (with harmonicity of V) in particular also implies that the three-dimensional Ricci-scalar vanishes,

$$R^{(3)} = 0\,. \tag{1.4}$$

[4] A lens space $L(p,q)$, with p, q coprime, is a quotient of the 3-sphere $|z_1|^2 + |z_2|^2 = 1$ defined by the identifications $(z_1, z_2) \sim (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$, with z_1, z_2 coordinates in $\mathbb{C}^2$.

Asymptotic flatness implies that one can, choosing appropriate coordinates, asymptotically write $g^{(3)}$ and V in the form

$$g^{(3)}_{ij} = \left(1 + \frac{2M}{r} + \mathcal{O}(r^{-2})\right)\delta_{ij}\,, \qquad V = 1 - \frac{M}{r} + \mathcal{O}(r^{-2})\,, \tag{1.5}$$

where M is the mass of the solution. Consider then the conformal transformations

$$(\Sigma, g^{(3)}) \to (\hat{\Sigma}_\pm, \hat{g}^{(3)}_\pm)\,, \qquad \hat{g}^{(3)}_\pm = \Omega^2_\pm g^{(3)} = \frac{1}{16}(1 \pm V)^4 g^{(3)}\,. \tag{1.6}$$

Then $\hat{g}^{(3)}_-$ can be extended to the point at infinity, and by a corollary of the positive mass theorem one can show that the union $\hat{\Sigma}_+ \cup \hat{\Sigma}_- \cup \{\infty\}$ (with $\hat{\Sigma}_\pm, \hat{g}^{(3)}_\pm$ continuously glued together at their boundaries) is isometric to $(\mathbb{R}^3, \delta)$. This in turn, in combination with the vacuum Einstein equations (1.1), implies spherical symmetry of the solution, which means the solution is indeed equivalent (isometric) to the Schwarzschild solution,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\,. \tag{1.7}$$

The proof generalises to asymptotically flat, static electrovacuum solutions with $|Q| < M$, with the unique solution being given by the Reissner–Nordström metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\,. \tag{1.8}$$

Note that this does not include the extremal case $|Q| = M$. In fact, in that case, another family of solutions is known to exist, which are the multi-black hole Majumdar–Papapetrou solutions. These reduce to the Reissner–Nordström solution if the number of centres $n = 1$.

Let us now get back to rotating (i.e. stationary rather than static) solutions.

Theorem 1.3. *Let $(\mathcal{M}, g)$ be an asymptotically flat, stationary and axisymmetric electrovacuum black hole solution with a non-degenerate event horizon. Then $(\mathcal{M}, g)$ is isometric to the Kerr–Newman solution with $a^2 + Q^2 < M^2$.*

To prove this more general statement, an argument of a rather different flavour is needed. The core of the argument is the following (see e.g. [69]). Given the assumptions set out in Theorem 1.3, one can show that the Killing fields corresponding to stationarity and axisymmetry satisfy integrability conditions that allow one to reduce the problem onto the two-dimensional orbit space defined as the quotient $\hat{\mathcal{M}} = \mathcal{M}/(\mathbb{R} \times U(1))$. The crucial observation then is that solutions of the resulting system can be mapped to solutions of a nonlinear sigma model. This is realised by mapping the gravitational data as well as the electric and magnetic field (with respect to a stationary observer) to the complex Ernst potentials E, Λ. The Ernst equations satisfied by (E, Λ) then describe a sigma model with target space given by the coset $SU(1,2)/S[U(1) \times U(2)]$. One can prove a uniqueness result for such a system, which finally translates to uniqueness of the resulting four-dimensional spacetime metric. This implies that the Kerr–Newman metric

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta - a^2\sin^2\theta}{\rho^2}\,\mathrm{d}t^2 + 2a\sin^2\theta\frac{\Delta - r^2 - a^2}{\rho^2}\,\mathrm{d}t\,\mathrm{d}\phi \\ &+ \sin^2\theta\left(r^2 + a^2 - a^2\sin^2\theta\frac{\Delta - r^2 - a^2}{\rho^2}\right)\mathrm{d}\phi^2 + \rho^2\left(\frac{\mathrm{d}r^2}{\Delta} + \mathrm{d}\theta^2\right) \end{aligned} \tag{1.9}$$

with

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta\,, \qquad \Delta(r) = r^2 - 2Mr + a^2 + Q^2\,, \qquad a^2 + Q^2 < M^2 \tag{1.10}$$

is the unique asymptotically flat, stationary and axisymmetric electrovacuum black hole solution. In fact, a version of Theorem 1.2 (with the additional assumption of axisymmetry) can be recovered as a special case of Theorem 1.3 by setting the angular momentum parameter $a = 0$.

Uniqueness theorems in higher dimensions

While we have seen that rotating black holes in higher dimensions are *not* in general uniquely specified by their conserved charges, in the static case the argument used to prove uniqueness of the Schwarzschild solution presented in section 1.1.1 can in fact be generalised to dimensions greater than four. This leads to the following result established in [57, 82].

Theorem 1.4. *Let $(\mathcal{M}, g)$ be an asymptotically flat, static vacuum black hole solution with a non-degenerate event horizon in dimension $D \geq 4$. Then $(\mathcal{M}, g)$ is isometric to the Schwarzschild–Tangherlini solution* [109],[5]

$$\mathrm{d}s^2 = -\left(1 - \frac{\mu}{r^{D-3}}\right)\mathrm{d}t^2 + \left(1 - \frac{\mu}{r^{D-3}}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{D-2}^2\,, \tag{1.11}$$

A similarly straightforward generalisation fails in the rotating case. Nonetheless, for the special case of five dimensions, and with some additional symmetry assumptions, a uniqueness result has been established. To formulate this result, a generalised definition of the orbit space (as used in the proof of uniqueness of Kerr–Newman) is needed.

Definition 1.1. *Let $(\mathcal{M}, g)$ be a D-dimensional black hole solution, $\mathcal{G}$ its group of isometries, and let $\langle\langle\mathcal{M}\rangle\rangle$ denote the domain of outer communication (DOC). Then $\hat{\mathcal{M}} = \langle\langle\mathcal{M}\rangle\rangle/\mathcal{G}$ is called the* orbit space *of $\mathcal{M}$.*

The following theorem concerning the structure of the orbit space has been established in [77].

[5] The mass parameter μ in (1.11) is related to the physical mass M of the solution via $\mu = \frac{16\pi G_D M}{(D-2)\Omega_{D-2}}$, such that in particular in four dimensions and with $G_4 = 1$, $\mu = 2M$.

Theorem 1.5. *Let $(\mathcal{M}, g)$ be a stationary, asymptotically flat or Kaluza–Klein vacuum black hole solution in D dimensions with isometry group $\mathcal{G} = \mathbb{R} \times U(1)^{D-3}$ and with Ricci tensor satisfying the null convergence condition*[6]*. Then the orbit space $\hat{\mathcal{M}}$ is a two-dimensional manifold with boundaries and corners homeomorphic to the upper half plane $\{(\rho, z) \mid \rho > 0\}$. Each segment of the boundary corresponds to either a component of the horizon, $\mathcal{H}/\mathcal{G}$, or an "axis" of the spacetime (a region where a linear combination of the rotational Killing fields vanishes).*

The latter part of this theorem means that one can think of the boundary $\partial\hat{\mathcal{M}}$ of the orbit space as a collection of intervals and vectors

$$\{(I_-, v_-), (I_1, v_1), \ldots, (I_{n-1}, v_{n-1}), (I_+, v_+)\} \tag{1.12}$$

with

$$\begin{aligned} I_- &\equiv I_0 \equiv (-\infty, z_1)\,, \\ I_i &\equiv (z_i, z_{i+1}) \ \text{ for } i = 1, \ldots, n-1\,, \\ I_+ &\equiv I_n \equiv (z_n, \infty)\,, \end{aligned} \tag{1.13}$$

and where the v_i (for $i = 0, \ldots, n$ with $v_- \equiv v_0$, $v_+ \equiv v_n$) denote the linear combination of the Killing vector fields vanishing on the respective interval. In case I_i corresponds to a horizon, no such combination vanishes, so v_i is just the zero vector. One can express the v_i as $(D-3)$-dimensional vectors of integers in the basis given by the rotational Killing vector fields $\{m_1, \ldots, m_{D-3}\}$. For two adjacent intervals I_i, I_{i+1}, if both I_i and I_{i+1} correspond to an axis, the vectors v_i, v_{i+1} have to satisfy the compatibility condition

$$\text{g.c.d.}\left\{\det\begin{pmatrix} v_i^K & v_i^L \\ v_{i+1}^K & v_{i+1}^L \end{pmatrix}\right\} = 1 \qquad \text{for } 1 \le K < L \le D-3\,, \tag{1.14}$$

where K, L are indices with respect to the basis of Killing vector fields $\{m_K\}$ and g.c.d. is the greatest common divisor.

Definition 1.2. *A collection of intervals and vectors of the form* (1.12)–(1.14) *is called the* rod structure [63] *of a solution.*

It is worth noting that in the special case $D = 5$, where there are two rotational Killing vector fields and hence the v_i can be represented as two-dimensional integer coefficient vectors, (1.14) simplifies to

$$\det\begin{pmatrix} \boldsymbol{v}_i \\ \boldsymbol{v}_{i+1} \end{pmatrix} = \pm 1\,. \tag{1.15}$$

With these ingredients, one can formulate a uniqueness theorem for five-dimensional black holes with $\mathbb{R} \times U(1)^2$-symmetry, as proved in [76].

[6] $R_{\mu\nu}\xi^\mu\xi^\nu \ge 0$ for any null vector ξ.

Theorem 1.6. *Let $(\mathcal{M}, g)$ be an asymptotically flat, stationary vacuum black hole solution in five dimensions with two commuting axial symmetries. Then $(\mathcal{M}, g)$ is, up to isometries, uniquely determined by its mass, its two angular momenta, and its rod structure.*

Some generalisations of this theorem have been established, e.g. to Einstein–Maxwell theory (given some additional assumptions) [75], to non-extremal solutions in minimal supergravity [3] or to eleven dimensional supergravity with eight rotational Killing fields [71]. However there are some fundamental differences of these uniqueness theorems to previously discussed results. Firstly, solutions are no longer specified by asymptotic charges only, but rather depend on some internal data. While this poses no complications from a mathematical point of view, it is not obvious how such a dependence could be interpreted in the context of string theory or the AdS/CFT correspondence—see e.g. [73] for a discussion. It also poses a challenge to the interpretation of microscopic entropy calculations in string theory [107] (this is discussed in more detail in chapter 4). Secondly, the statement of the theorem does not touch the important question of existence. It is unclear, which rod structures (and in combination with which ranges of charges) lead to physical (in particular, stably causal, regular on and outside the horizon) solutions.

1.1.2 Known solutions

In the light of the above discussed uniqueness theorems, this section gives a brief description of some known higher-dimensional black hole solutions, with a focus on five dimensions. We will restrict to non-supersymmetric solutions here, with supersymmetric solutions being discussed separately in section 1.2.

Myers–Perry solutions

These solutions, found by Myers and Perry in the 1980s [94] (see e.g. [44, 93] for overviews), are the natural generalisation of the Kerr black hole—i.e. stationary, asymptotic flat, vacuum solutions—to arbitrary dimension $D > 4$. In the most general case, solutions will rotate in $N = [\frac{D-1}{2}]$ independent rotation planes. As a consequence, to write down the general form of the solution, one has to distinguish between odd ($D = 2N + 1$) and even ($D = 2N + 2$) number of dimensions. For simplicity we will here focus on the five-dimensional solution only. For $D = 5$, the spacetime possesses $N = 2$ orthogonal rotation planes, and the metric takes the form

$$\begin{aligned} \mathrm{d}s^2 = -\,\mathrm{d}t^2 + \frac{\mu}{\Sigma}\left(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi - b\cos^2\theta\,\mathrm{d}\psi\right)^2 + \frac{r^2\Sigma}{\Delta}\,\mathrm{d}r^2 \\ + \left(r^2 + a^2\right)\sin^2\theta\,\mathrm{d}\phi^2 + \left(r^2 + b^2\right)\cos^2\theta\,\mathrm{d}\psi^2 + \Sigma\,\mathrm{d}\theta^2\,, \end{aligned} \tag{1.16}$$

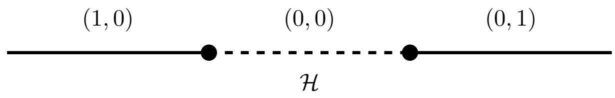


Figure 1.2: Rod diagram for the five-dimensional Myers–Perry solution. The vectors vanishing on each interval are given with respect to the basis $\{\partial_\phi, \partial_\psi\}$.

where ϕ and ψ are 2π-periodic, $\theta \in [0, \pi/2]$ and

$$\Sigma(r,\theta) = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \qquad \Delta(r) = (r^2+a^2)(r^2+b^2) - \mu r^2\,, \tag{1.17}$$

with a, b and μ parameters of the solution. The asymptotic behaviour of the metric determines the mass and two angular momenta of the solution as

$$M = \frac{3\pi}{8}\mu\,, \qquad J_\phi = \frac{\pi}{4}\mu a\,, \qquad J_\psi = \frac{\pi}{4}\mu b\,, \tag{1.18}$$

and the solution is regular with a horizon at

$$r_0^2 = \frac{1}{2}\left(\mu - a^2 - b^2 + \sqrt{(\mu - a^2 - b^2)^2 - 4a^2b^2}\right)\,, \tag{1.19}$$

provided that $(\mu - a^2 - b^2)^2 - 4a^2b^2 \geq 0$, or equivalently in terms of the charges

$$\big(|J_\phi| + |J_\psi|\big)^2 \leq \frac{32}{27\pi}M^3\,. \tag{1.20}$$

Saturation of the inequality corresponds to the extremal limit of the solution. One can express (1.20) more conveniently in terms of the reduced angular momenta,

$$\eta_i \equiv \sqrt{\frac{27\pi}{32}}\frac{J_i}{M^{3/2}}\,, \qquad i = \phi, \psi \tag{1.21}$$

as

$$\big(|\eta_\phi| + |\eta_\psi|\big)^2 \leq 1\,. \tag{1.22}$$

The horizon at $r = r_0$ has topology S^3 and area given by

$$A_H = 2\pi^2\mu r_0\,. \tag{1.23}$$

Finally, the rod diagram of the five-dimensional Myers–Perry black hole is depicted in Figure 1.2. By Theorem 1.6, the charges (1.18) together with the rod structure uniquely specify the solution.

Black Rings

Black rings are asymptotically flat black hole solutions which have horizon topology $S^1 \times S^2$. They were first constructed as vacuum solutions in five dimensions in [41]. In their original form, solutions have one angular momentum, associated to a rotation along the S^1 direction of the ring. More general black ring solutions have since been found. This includes, among others, solutions with two angular momenta [99], charged black rings [36], or supersymmetric black rings [37] which are discussed in more detail in section 1.2.3. Moreover, explicit examples have also been constructed of multi-black ring solutions [50, 51], or a bound state of a spherical black hole and a black ring, dubbed a “black saturn” [39]. An extensive review of black ring solutions can be found e.g. in [43]. We will here focus on the simplest case, that of a black ring rotating solely the S^1 direction, hence carrying a mass and one angular momentum J (“singly spinning black ring”). The metric of the solution can be given as [40]

$$\mathrm{d}s^2 = -\frac{F(y)}{F(x)}\left(\mathrm{d}t + \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}R\frac{1+y}{F(y)}\,\mathrm{d}\psi\right)^2 \\ + \frac{R^2}{(x-y)^2}F(x)\left(-\frac{G(y)}{F(y)}\,\mathrm{d}\psi^2 - \frac{\mathrm{d}y^2}{G(y)} + \frac{\mathrm{d}x^2}{G(x)} + \frac{G(x)}{F(x)}\,\mathrm{d}\phi^2\right)\,, \quad (1.24)$$

where

$$F(\xi) = 1 + \lambda\xi\,, \qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \quad (1.25)$$

and the parameters

$$0 < \nu \leq \lambda < 1\,, \qquad R > 0\,. \quad (1.26)$$

In particular, the parameter R corresponds to the radius of the S^1 of the black ring, and the coordinates (x, y) are related to double polar coordinates (r_1, ϕ), (r_2, ψ) on $\mathbb{R}^4$ as

$$x = \frac{R^2 - r_1^2 - r_2^2}{\Sigma}\,, \qquad y = -\frac{R^2 + r_1^2 + r_2^2}{\Sigma}\,, \quad (1.27)$$

with

$$\Sigma = \sqrt{(R^2 + r_1^2 + r_2^2)^2 - 4R^2 r_2^2}\,. \quad (1.28)$$

This implies the ranges

$$-1 \leq x \leq 1\,, \qquad -\infty < y \leq -1\,. \quad (1.29)$$

From (1.24) it is easy to see that $g_{\phi\phi}$, $g_{\psi\psi}$ vanish if $x = \pm 1$, $y = -1$, respectively. In order to avoid conical singularities, the periods of ϕ and ψ have to be fixed to

$$\phi \sim \phi + 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\,, \qquad \psi \sim \psi + 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\,. \quad (1.30)$$

and (λ, ν) must further satisfy

$$\lambda = \frac{2\nu}{1+\nu^2}\,, \qquad 0 < \nu < 1\,. \quad (1.31)$$

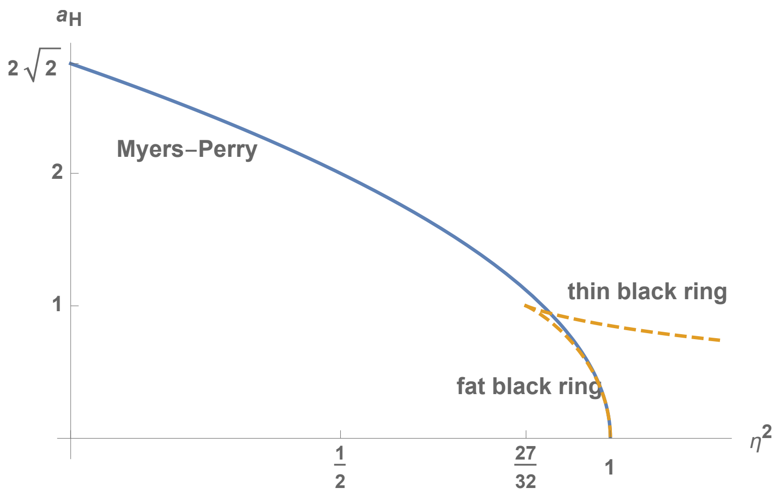


Figure 1.3: Reduced area of the Myers–Perry (blue/solid) and black ring (orange/dashed) solutions.

With these choices, the solution is regular on and outside a horizon at $y = -1/\nu$ of topology $S^1 \times S^2$. The horizon area is given by

$$A_H = 8\pi^2 R^3 \frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)}\,, \tag{1.32}$$

and the mass and angular momentum of the solution can be obtained from the asymptotic expansion of (1.24) (spatial infinity corresponds to $(x, y) \to (-1, -1)$) as

$$M = \frac{3\pi R^2}{4}\frac{\lambda}{1-\nu}\,, \qquad J = \frac{\pi R^3}{2}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2}\,. \tag{1.33}$$

It is important to note that (1.31) implies a lower bound on the reduced angular momentum,

$$\eta \equiv \sqrt{\frac{27\pi}{32}}\frac{J}{M^{3/2}} = \sqrt{\frac{(1+\nu)^3}{8\nu}} \geq \sqrt{\frac{27}{32}}\,. \tag{1.34}$$

This is different to what was seen for the Myers–Perry solution, where regularity imposed an upper bound on the two reduced angular momenta, (1.22). In particular, one can compare (1.34) to a Myers–Perry solution with one non-vanishing angular momentum, say $\eta_\phi \equiv \eta$, $\eta_\psi = 0$. Then, for reduced angular momentum in the range

$$\sqrt{\frac{27}{32}} \leq \eta < 1 \tag{1.35}$$

both, Myers–Perry and black ring solutions exist. As pointed out earlier, this example was the first explicit evidence for black hole non-uniqueness in higher dimensional spacetimes. It is interesting to compare the horizon areas of the two distinct solutions

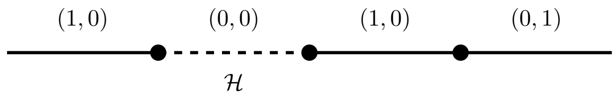


Figure 1.4: Rod diagram for the black ring solution. The distinction between different types of black ring solutions (fat vs. thin black rings) is reflected in the interval lengths, which are not specified in this depiction.

in the region of overlap. Defining a dimensionless area,

$$a_H = \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{A_H}{M^{3/2}}\,, \tag{1.36}$$

for the two solutions one finds

$$a_H^{\mathrm{MP}} = 2\sqrt{2(1-\eta^2)}\,, \qquad a_H^{\mathrm{ring}} = 2\sqrt{\nu(\eta)(1-\nu(\eta))}\,, \tag{1.37}$$

where in the ring case $\nu(\eta)$ is defined by the relation (1.34). Figure 1.3 shows a comparison of the two areas. Note that the black ring solutions splits into two branches: *thin* black rings ($0 < \nu < 1/2$), and *fat* black ring ($1/2 < \nu < 1$). One can see that the thin black ring solution has horizon area greater than that of the Myers–Perry solution for some range of ν in the region of overlap, while the area of the fat black ring always stays below that of Myers–Perry.

Finally, the rod structure for black ring solutions is depicted in Figure 1.4.

1.1.3 Black hole thermodynamics

A major milestone in understanding black holes was the realisation in the 1970s that black holes satisfy a set of rules similar to the zeroth, first and second law of thermodynamics [6]. In a simplified form[7], these *laws of black hole mechanics* are:

0. For a stationary black hole spacetime, the surface gravity κ is constant on the horizon.

1. Upon perturbations,
$$\delta M = \frac{\kappa}{8\pi}\delta A_{\mathcal{H}} + \Omega\delta J\,. \tag{1.38}$$

2. The rate of change of the area of the horizon in time is nonnegative.

In particular the striking resemblance between the roles played by entropy in the thermodynamical context and the horizon area for black holes led to two proposals by Bekenstein: First, that in fact the horizon area of the black hole should be interpreted as its entropy (up to a multiplicative factor) [8], and second, that nature follows a *generalised second law*: In a spacetime containing a black hole, the sum of the entropies of the black hole and the exterior does not decrease [9].

[7] See e.g. [118] for the precise statements and a detailed discussion.

Most famously around the same time an addition to the purely classical arguments above was provided by Hawking [65], who discovered that as a result of quantum effects near the horizon, black holes evaporate, where energy is emitted through radiation following a black body spectrum of temperature

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}\,, \tag{1.39}$$

known as the Hawking temperature of the black hole. Together with the proposed relation $S \sim A_{\mathcal{H}}$, this allows one to identify

$$S_{\mathrm{BH}} = \frac{A_{\mathcal{H}}}{4G}\,. \tag{1.40}$$

This is known as the Bekenstein–Hawking entropy of a black hole.

While we have seen that many properties of four-dimensional black holes break down in higher spacetime dimensions, the laws of black hole mechanics are in general valid in any dimension [44]. However, due to additional features of black holes that are not present in four spacetime dimensions, the first law (1.38) may have to be modified. This can be done straightforwardly to incorporate additional charges of the black hole, however it might also include terms not related to conserved charges (such as dipole charges). This, for example, is indeed the case for the black ring [4, 29, 40]. For a generalised treatment of the first law in higher dimensions see [87].

Similarly, although explicit computations are more challenging in general, the concept of Hawking radiation and thus Hawking temperature (1.39) and the expression for the Bekenstein Hawking entropy (1.40) carry over to higher dimensions.

1.2 Preliminaries II: Black holes in String Theory

The previous chapter has dealt with classical solutions of general relativity. It is clear that a full understanding of black holes can only be gained in the context of a theory of quantum gravity. We will thus go on to study black holes in the context of string theory.

1.2.1 Black holes from branes

An important additional feature of string theory is that its natural "black" objects are not necessarily black holes in the standard sense, but will in general be extended objects: black branes. This is due to the presence of general n-form potentials: In the same manner as in four dimensions the electromagnetic 1-form gauge potential $A^{(1)}$ couples to a point particle, a general n-form potential $A^{(n)}$ couples to an object of $(n-1)$ spatial dimensions electrically, or an object of $(D-n-3)$ spatial dimensions magnetically (via the dual field strength). Thus it is these higher-dimensional objects, or branes, that naturally carry charge under $A^{(n)}$. A black p-brane is then an object that is extended in p spatial dimensions, is translationally invariant in its internal directions, and has a black hole geometry in the remaining directions (see e.g. [97]). What types of branes are present in a given theory is determined by its field content.

This work is concerned with asymptotically flat black hole solutions. By construction, nontrivial p-brane solutions (with the possible exception of $p = 0$) are not asymptotically flat in the full-dimensional spacetime. However, systems of intersecting branes, provide a framework to create asymptotically flat black hole solutions in a lower dimensional theory via dimensional reduction. To see this, let us first briefly discuss intersecting brane solutions. For the sake of simplicity we will do this in M-theory; brane solutions to type IIA/B string theory can easily be obtained from there via dimensional reduction and T-duality. We will furthermore restrict to solutions that preserve some supersymmetry, i.e. they saturate the Bogomol'ny–Prasad–Sommerfield (BPS) bound, see e.g. [105]. The low energy effective action of M-theory is that of eleven-dimensional supergravity [31], with bosonic part (see e.g. [49])

$$S_{\rm bos} = \frac{1}{2\kappa^2}\int (R \star 1 - \frac{1}{2}F^{(4)} \wedge \star F^{(4)} - \frac{1}{6}A^{(3)} \wedge F^{(4)} \wedge F^{(4)})\,, \tag{1.41}$$

where $A^{(3)}$ is a 3-form gauge field and $F^{(4)} = \mathrm{d}A^{(3)}$. Variation of (1.41) yields the field equations

$$R_{\mu\nu} = \frac{1}{12}(F_{\mu\rho\sigma\lambda}F_\nu{}^{\rho\sigma\lambda} - \frac{1}{12}g_{\mu\nu}F^2)\,, \qquad \mathrm{d}\star F + \frac{1}{2}F \wedge F = 0\,. \tag{1.42}$$

The 4-form field strength $F^{(4)}$ will naturally couple to $(2+1)$-dimensional M2-branes or $(5+1)$-dimensional M5-branes. The solution for a BPS M2-brane stretching in the x_1 and x_2 directions can be written in the simple form [13, 35] (see e.g. [7] for an overview)

$$\mathrm{d}s^2 = H_2^{-2/3}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2) + H_2^{1/3}(\mathrm{d}x_3^2 + \ldots + \mathrm{d}x_{10}^2)\,, \tag{1.43}$$

$$F^{(4)} = \mathrm{d}t \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}H_2^{-1}\,, \tag{1.44}$$

where H_2 is a harmonic function on the orthogonal $\mathbb{R}^8$. Similarly, for the BPS M5-brane [59]

$$\mathrm{d}s^2 = H_5^{-1/3}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_5^2) + H_5^{2/3}(\mathrm{d}x_6^2 + \ldots + \mathrm{d}x_{10}^2)\,, \tag{1.45}$$

$$F^{(4)} = \star(\mathrm{d}t \wedge \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^5 \wedge \mathrm{d}H_5^{-1})\,, \tag{1.46}$$

where now H_5 is harmonic on the $\mathbb{R}^5$ with coordinates $x_6, \ldots, x_{10}$.

One can easily obtain multi-brane solutions by application of the *harmonic function rule* [117]. For example, when considering an M2-brane and an M5-brane both with directions as above (i.e. the branes are parallel in the x_1 and x_2 direction), for the intersecting brane solution one may use an ansatz (see e.g. [48])

$$\begin{aligned} \mathrm{d}s^2 = {} & H_2^{-2/3} H_5^{-1/3} (-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2) \\ & + H_2^{1/3} H_5^{-1/3} (\mathrm{d}x_3^2 + \cdots + \mathrm{d}x_5^2) + H_2^{1/3} H_5^{2/3} (\mathrm{d}x_6^2 + \ldots + \mathrm{d}x_{10}^2) \end{aligned} \tag{1.47}$$

where now the harmonic functions only depend on the shared orthogonal directions, such that the M2-brane is *smeared* over directions $x_3, \ldots, x_4$. The field strength $F^{(4)}$ is simply given by the sum of the separate solutions (1.44), (1.46) (albeit with dependencies of the harmonic functions as in (1.47)). This system can be summarised graphically as

$$\begin{array}{c|ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \mathrm{M2} & - & - & - & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot & \cdot \\ \mathrm{M5} & - & - & - & - & - & - & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{1.48}$$

where "$\sim$" marks smeared directions.

All solutions presented here are BPS solutions, so they preserve some supersymmetry. Each of the M2- and M5-brane solutions, (1.43) and (1.45) preserves precisely $1/2$, the intersection of the two (1.47) preserves $1/4$ of the supersymmetry. Generically, an intersection of n BPS branes will preserve $1/2^n$ of the supersymmetry (although exceptions exist, see [48] and references therein).

The generic choice for the harmonic functions in (1.43), (1.45) are harmonic functions with a single pole, $H_2 = 1 + r_2^6/r^6$ for the M2-brane and $H_5 = 1 + r_2^3/r^3$ for the M5-brane. The constants r_2, r_5 can be related to the mass per unit volume of the branes via the asymptotic expansion of g_{tt}, see e.g. [7]. Generalising this to a multi-centred ansatz leads to stacks of parallel branes at different positions, see e.g. [48].

As has been pointed out earlier, one may obtain solutions to type IIA/B supergravity from the eleven-dimensional theory. Type IIA supergravity arises as Kaluza–Klein compactification of eleven-dimensional supergravity on a circle, with ansatz

$$\mathrm{d}s_{11}^2 = e^{\frac{4}{3}\Phi} (\mathrm{d}y + C^{(1)})^2 + e^{-\frac{2}{3}\Phi}\,\mathrm{d}S_{10}^2\,, \qquad A^{(3)} = B^{(2)} \wedge \mathrm{d}y + C^{(3)}\,, \tag{1.49}$$

where Φ is the dilaton field and $C^{(1)}$, $B^{(2)}$ and $C^{(3)}$ are 1-, 2- and 3-forms on the ten-dimensional base respectively. We have used a capital letter for the ten-dimensional metric $\mathrm{d}S_{10} = G_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ to emphasise this is the metric in the string frame, that is, the Einstein–Hilbert part of the action includes the dilaton, $S \sim \int e^{-2\Phi} R^{(G)} \star_{(G)} 1$.[8] Consider as an example the M2-brane solution (1.43) and compactify along x^{10}. Then clearly the dilaton $e^\Phi = H_2^{1/4}$ and one finds

$$\mathrm{d}S_{10} = H_2^{-1/2}(-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2) + H_2^{1/2}(\mathrm{d}x_3^2 + \ldots + \mathrm{d}x_9^2)\,, \tag{1.50}$$

which describes a D2-brane in type IIA supergravity. Similarly the NS5-brane of type

[8] $G_{\mu\nu}$ is related to the Einstein frame metric $g_{\mu\nu}$ with $S \sim \int R^{(g)} \star 1$ via a conformal transformation $g_{\mu\nu} = e^{-4\Phi/(D-2)} G_{\mu\nu} = e^{-\Phi/2} G_{\mu\nu}$.

IIA can be obtained upon compactifying (1.45) along (say) x^{10}, yielding

$$\mathrm{d}S_{10} = (-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \ldots \mathrm{d}x_5^2) + H_5(\mathrm{d}x_6^2 + \ldots + \mathrm{d}x_9^2) \tag{1.51}$$

with dilaton $e^{\Phi} = H_5^{1/2}$. A little less straightforwardly[9], the D6-brane solution of type IIA can be found as

$$\mathrm{d}S_{10} = V^{-1/2}(-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + \ldots \mathrm{d}x_6^2) + V^{1/2}(\mathrm{d}x_7^2 + \ldots + \mathrm{d}x_9^2)\,, \tag{1.52}$$

where V is harmonic on $\mathbb{R}^3$ and the dilaton is $e^{\Phi} = V^{-3/4}$. Type IIB solutions can then be obtained through T-duality. As for the purpose of this work we are merely interested in branes as the building blocks of black hole solutions, will not go into further details here (these may be found e.g. in [48]). Instead we will move on to give an explicit example of constructing a black hole solution from branes.

Example: The extremal Reissner–Nordström black hole

As an illustrative example, let us reproduce the extremal Reissner–Nordström metric starting from a system of branes (we will mostly follow [96] for this purpose). Consider the following system of branes in type IIA:

$$\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathrm{D2} & - & - & - & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot \\ \mathrm{D6} & - & - & - & - & - & - & - & \cdot & \cdot & \cdot \\ \mathrm{NS5} & - & - & - & - & - & - & \sim & \cdot & \cdot & \cdot \\ \mathrm{W} & - & \to & \sim & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot \end{array} \tag{1.53}$$

Here W denotes a wave carrying some momentum in the direction of x_1. An ansatz for the metric can be found using again the harmonic function rule exemplified in (1.47). This leads to a metric

$$\begin{aligned} \mathrm{d}S_{10}^2 = {} & H_2^{-1/2}H_6^{-1/2}(-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + K(\mathrm{d}t + \mathrm{d}x_1)^2 + \mathrm{d}x_2^2) \\ & + H_2^{1/2}H_6^{-1/2}(\mathrm{d}x_3^2 + \mathrm{d}x_4^2 + \mathrm{d}x_5^2) + H_2^{1/2}H_6^{-1/2}H_5\,\mathrm{d}x_6^2 + H_2^{1/2}H_6^{1/2}H_5(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2)\,, \end{aligned} \tag{1.54}$$

where H_2, H_5 and H_6 are the harmonic functions related to the D2-, NS5- and D6-branes respectively, and K is an additional harmonic function related to the wave W. The dilaton field Φ is given by

$$e^{\Phi} = H_2^{1/4}H_6^{-3/4}H_5^{1/2}\,. \tag{1.55}$$

We will take a multi-centred ansatz for the harmonic functions (r is the radius on

[9] In this case one needs to start from a Kaluza–Klein monopole in eleven dimensions, see [115].

the orthogonal $\mathbb{R}^3$)

$$H_i = 1 + \frac{r_i}{r}\,, \qquad K = \frac{r_m}{r}\,. \tag{1.56}$$

Compactifying directions $x_2, \ldots, x_6$ results in the five-dimensional metric

$$\mathrm{d}S_5^2 = H_2^{-1/2} H_6^{-1/2} (-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + K(\mathrm{d}t + \mathrm{d}x_1)^2) + H_2^{1/2} H_6^{1/2} H_5 (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2) \tag{1.57}$$

with dilaton

$$e^{\Phi_5} = e^{\Phi} H_2^{-3/8} H_6^{5/8} H_5^{-1/4} = H_2^{-1/8} H_6^{-1/8} H_5^{1/4}\,, \tag{1.58}$$

determined by fixing the prefactor in the five-dimensional Einstein–Hilbert action accordingly. For the final compactification step, write (1.57) as

$$\begin{aligned} \mathrm{d}S_5^2 = -H_2^{-1/2} H_6^{-1/2} (1+K)^{-1}\,\mathrm{d}t^2 &+ H_2^{1/2} H_6^{1/2} H_5 \left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\right) \\ &+ H_2^{-1/2} H_6^{-1/2} (1+K) \left(\mathrm{d}x_1 + \frac{K}{1+K}\,\mathrm{d}t\right)^2 . \end{aligned} \tag{1.59}$$

The four-dimensional string frame metric after compactifying along x_1 is precisely the first line of (1.59). One can obtain the metric in the Einstein frame by multiplying with the appropriate power of the dilaton given by

$$e^{\Phi_4} = e^{\Phi_5} H_2^{1/8} H_6^{1/8} (1+K)^{-1/4} = H_5^{1/4} (1+K)^{-1/4}\,, \tag{1.60}$$

as

$$\mathrm{d}s_4^2 = e^{-2\Phi_4}\,\mathrm{d}S_4^2 = -[H_2 H_6 H_5 (1+K)]^{-1/2}\,\mathrm{d}t^2 + [H_2 H_6 H_5 (1+K)]^{1/2} (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2)\,. \tag{1.61}$$

For general r_i in the harmonic functions, this gives a four-charge black hole with mass

$$M = \frac{1}{4G_4} (r_2 + r_6 + r_5 + r_m)\,. \tag{1.62}$$

Setting $G_4 = 1$ and choosing $r_2 = r_6 = r_5 = r_m = Q$ one recovers the metric of the extremal Reissner–Nordström black hole,[10]

$$\mathrm{d}s_{\mathrm{ERN}}^2 = -\left(1 + \frac{Q}{r}\right)^{-2} \mathrm{d}t^2 + \left(1 + \frac{Q}{r}\right)^2 (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2)\,. \tag{1.63}$$

The Strominger–Vafa black hole and microscopic derivation of black hole entropy

We have seen that black holes can be modelled as brane configurations in string theory. This has in fact led to one of the most significant successes of string theory: a microscopic derivation of the Bekenstein–Hawking entropy from the degeneracy of states in the brane system [107]. Let us first describe the black hole solution. Consider a system

[10] The metric presented here is related to (1.8) with $M = Q$ via a coordinate change $r \to r - Q$.

similar to the one in the previous section in type IIB,[11]

$$\begin{array}{lcccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \mathrm{D1} & - & - & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot \\ \mathrm{D5} & - & - & - & - & - & - & \cdot & \cdot & \cdot & \cdot \\ \mathrm{W} & - & \rightarrow & \sim & \sim & \sim & \sim & \cdot & \cdot & \cdot & \cdot \end{array} \tag{1.64}$$

The ten-dimensional metric for this system is

$$\begin{aligned} \mathrm{d}S_{10}^2 = H_1^{-1/2} H_5^{-1/2} (-\,\mathrm{d}t^2 + \mathrm{d}x_1^2 + K(\mathrm{d}t + \mathrm{d}x_1)^2) \\ + H_1^{1/2} H_5^{-1/2} (\mathrm{d}x_2^2 + \ldots + \mathrm{d}x_5^2) + H_1^{1/2} H_5^{1/2} (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2) \end{aligned} \tag{1.65}$$

with dilaton

$$e^{\Phi} = H_1^{1/2} H_5^{-1/2} \tag{1.66}$$

and the harmonic functions are again chosen to be one-centred, however this time on $\mathbb{R}^4$, so

$$H_i = 1 + \frac{r_i^2}{r^2}\,, \qquad K = \frac{r_m^2}{r^2}\,. \tag{1.67}$$

In the same manner as for the reduction to the four-dimensional extremal Reissner–Nordström solution above, upon compactifying directions $x_1, \ldots, x_5$, one obtains the five-dimensional black hole solution

$$\mathrm{d}S_5^2 = -(H_1 H_5)^{-1/2} (1+K)^{-1}\,\mathrm{d}t^2 + (H_1 H_5)^{1/2} (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2) \tag{1.68}$$

with dilaton

$$e^{\Phi_5} = H_1^{1/8} H_5^{1/8} (1+K)^{-1/4}\,, \tag{1.69}$$

so the metric in the Einstein frame is given by

$$\mathrm{d}s_5^2 = -[H_1 H_5 (1+K)]^{-2/3}\,\mathrm{d}t^2 + [H_1 H_5 (1+K)]^{1/3} (\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2)\,. \tag{1.70}$$

This is an extremal, non-rotating, three-charge black hole in five dimensions with mass

$$M = \frac{\pi}{4G_5} (r_1^2 + r_5^2 + r_m^2)\,. \tag{1.71}$$

One may relate the parameters r_1, r_5 and r_m to integer-quantised charges Q_1, Q_5, N as (see e.g. [33])

$$r_1^2 = \frac{4 G_5 R_1}{\pi \alpha' g_s} Q_1\,, \qquad r_5^2 = \alpha' g_s Q_5\,, \qquad r_m^2 = \frac{4 G_5}{\pi R_1} N\,, \tag{1.72}$$

[11] This particular configuration may be derived from a system of 3 M2-branes in eleven dimensions, see e.g. [33]. The relevant powers of the harmonic functions in the metric may be found in e.g. [48]. Note that the system presented here is slightly more general then the one originally considered by Strominger and Vafa in [107].

where R_1 is the compactification radius along the x_1 direction of (1.64). The black hole (1.70) has a horizon at $r = 0$ with area

$$A_{\mathcal{H}} = 2\pi^2 r_1 r_5 r_m \,, \tag{1.73}$$

so using (1.72) one finds the Bekenstein–Hawking entropy of the black hole to be given by

$$S_{\mathrm{BH}} = \frac{A_{\mathcal{H}}}{4G_5} = 2\pi\sqrt{Q_1 Q_5 N}\,. \tag{1.74}$$

From a ten-dimensional viewpoint, the entropy of the system can be computed from the degeneracy of states. It has be shown that the low energy effective theory of the D1-D5-W system in fact reduces to a two-dimensional supersymmetric gauge theory, see e.g. [89]. This can be seen from studying the massless modes of strings between the branes (with the simplifying assumption that the compactification radius R_1 is significantly larger than the compactification radii of the T^4 in directions $x_2, \ldots, x_5$). In particular the bound brane state can be found to correspond to a two-dimensional SCFT with central charge $c = 6Q_1Q_5N$, such that by the Cardy formula for the entropy of a two-dimensional CFT [23] one recovers the entropy (1.74).

The BMPV solution

The results of [107] were generalised to a 2-parameter family of spinning black holes shortly thereafter [15], and further to a 4-parameter family (similar to the three-charge version of the Strominger–Vafa black hole presented in the previous section) in [116]. This is called the Breckenridge–Myers–Peet–Vafa (BMPV) solution. In its generalised form, its metric is given by[12]

$$\mathrm{d}s_5^2 = -[H_1H_5(1+K)]^{-2/3}(\mathrm{d}t+\omega)^2 + [H_1H_5(1+K)]^{1/3}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_3^2)\,, \tag{1.75}$$

with

$$\omega = \frac{a}{r^2}\left(\sin^2\theta\,\mathrm{d}\phi_1 + \cos^2\theta\,\mathrm{d}\phi_2\right)\,, \tag{1.76}$$

where (θ, ϕ_1, ϕ_2) parametrise the 3-sphere with metric $\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi_1^2 + \cos^2\theta\,\mathrm{d}\phi_2^2$. The harmonic functions are as above and the mass is again given by (1.71). The new solution however has two equal, non-vanishing angular momenta,

$$J_1 = J_2 = \frac{\pi}{4G_5}a\,. \tag{1.77}$$

[12] The original 2-parameter solution was in fact constructed differently, via uplifting the Myers–Perry solution to six dimensions, performing boosts as well as exploiting string dualities in the six-dimensional framework, and then reducing back to a new, five-dimensional solution. In any case, we will here directly generalise the brane ansatz (1.70), following [116].

From (1.75) one can see that the spacetime has a horizon at $r = 0$, with area

$$A_{\mathcal{H}} = 2\pi^2 (r_1^2 r_5^2 r_m^2 - a^2)^{1/2}\,, \tag{1.78}$$

such that

$$S_{\rm BH} = 2\pi \sqrt{Q_1 Q_5 N - a^2}\,. \tag{1.79}$$

The original BMPV solution of [15] can be recovered through the choice

$$H_1 = H_5 = 1 + K = 1 + \mu/r^2\,, \tag{1.80}$$

as

$$\begin{aligned} \mathrm{d}s_5^2 = -\left(1+\frac{\mu}{r^2}\right)^{-2} &\left[\mathrm{d}t + \frac{a}{r^2}\left(\sin^2\theta\,\mathrm{d}\phi_1 + \cos^2\theta\,\mathrm{d}\phi_2\right)\right]^2 \\ &+ \left(1+\frac{\mu}{r^2}\right)\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi_1^2 + \cos^2\theta\,\mathrm{d}\phi_2^2)\right]\,. \end{aligned} \tag{1.81}$$

with mass $M = 3\pi\mu/4G_5$ and angular momenta unchanged, and

$$S_{\rm BH} = 2\pi^2 \sqrt{\mu^3 - a^2}\,. \tag{1.82}$$

We have constructed some examples of extremal black holes via Kaluza–Klein reduction of a system of branes in ten (or eleven) dimensions. We shall in the following section see that the five-dimensional solutions presented do in fact represent solutions to minimal supergravity in five dimensions.

1.2.2 Minimal supergravity in five dimensions

The classification of supergravity solutions, in particular of such which preserve some (or all) supersymmetry, has been a field of intense study since the development of supergravity. As a result, general supersymmetric solutions to many theories are now known. This includes in particular theories with a low number of supercharges see e.g. [58]. In general, classifying supersymmetric solutions involves solving the field equations as well as the Killing spinor equations—conditions that arise from imposing vanishing variation of the fermionic fields under supersymmetry transformations. These are local constraints, so classifications are typically of a purely local nature.

For minimal $\mathcal{N} = 1$ supergravity in $D = 5$ dimensions, as constructed in [30], supersymmetric solutions have been studied and classified in [53]. The bosonic field content of five-dimensional minimal supergravity is given by a metric g and a Maxwell field $F = \mathrm{d}A$ with action

$$S_{\rm bos} = S_{EH} + S_{Maxwell} + S_{CS} = \frac{1}{2\kappa_5^2}\int\left(\star R - 2F\wedge\star F - \frac{8}{3\sqrt{3}}F\wedge F\wedge A\right)\,. \tag{1.83}$$

Variation with respect to g and A yields the equations of motion

$$R_{\mu\nu} = 2(F_{\mu\rho}F^{\nu\rho} - \frac{1}{3}g_{\mu\nu}F^2)\,, \qquad \mathrm{d}\star F + \frac{2}{\sqrt{3}}F\wedge F = 0\,. \tag{1.84}$$

A solution is supersymmetric if it admits a Killing spinor, i.e. a spinor that is constant with respect to the supercovariant derivative,

$$\mathcal{D}_\mu \epsilon^a = 0\,, \tag{1.85}$$

where the exact form of $\mathcal{D}_\mu$ depends on the Maxwell field F and ϵ^a is a symplectic Majorana spinor, see [53] for conventions. In five dimension, such a spinor has 8 real (or 4 complex) components. These 8 degrees of freedom can be expressed in terms of a function f, a 1-form V and three 2-forms $\Phi^{ab} \equiv \Phi^{(ab)}$ by constructing the spinor bilinears

$$f\epsilon^{ab} = i\bar{\epsilon}^a\epsilon^b\,, \tag{1.86}$$

$$V_\mu\epsilon^{ab} = \bar{\epsilon}^a\gamma_\mu\epsilon^b\,, \tag{1.87}$$

$$\Phi^{ab}_{\mu\nu} = i\bar{\epsilon}^a\gamma_{\mu\nu}\epsilon^b\,, \tag{1.88}$$

where f and V are real, Φ^{12} is purely imaginary and $\Phi^{11} = (\Phi^{22})^*$. These are not all independent, but algebraically related via Fierz identities, such that after exploiting these the remaining degrees of freedom of the newly obtained quantities match the eight degrees of freedom of ϵ.

What is important to notice is that as ϵ is a Killing spinor, V will be a Killing vector of the solution (g, F). Furthermore one finds

$$V_\mu V^\mu = -f^2 \leq 0\,, \tag{1.89}$$

such that V will be either null or timelike.[13] It has been shown in [53] that if V is null, solutions are plane-fronted waves. As such they will not be asymptotically flat in the five-dimensional sense (unless trivial), such that we will in the following focus on the case of timelike V. In that case, by choosing a time coordinate t such that $V = \partial/\partial t$, it is possible to locally write the spacetime metric in the form

$$\mathrm{d}s^2 = -f^2(\mathrm{d}t+\omega)^2 + f^{-1}h\,, \tag{1.90}$$

where the base space metric h is hyper-Kähler, i.e. it is Riemannian with three compatible complex structures $\{I, J, K\}$ that satisfy the quaternionic identities $I^2 = J^2 = K^2 = IJK = -1$ (see e.g. [70]), and where f and ω are a function and 1-form on this base space respectively.

A particularly interesting subclass of solutions are those for which the base space

[13] The case $V = 0$ can be excluded, see [53].

manifold is a Gibbons–Hawking space [54]. In that case there is an additional $U(1)$-symmetry in the base space, and its metric takes the form

$$h = H^{-1}(\mathrm{d}\psi + \chi)^2 + H\,\mathrm{d}x^i\,\mathrm{d}x^i\,, \tag{1.91}$$

where ∂_ψ is the $U(1)$-Killing vector, the x^i are Cartesian coordinates of three-dimensional Euclidean space $\mathbb{R}^3$, H is a harmonic function on this space and χ a 1-form on $\mathbb{R}^3$ satisfying

$$\star_3\,\mathrm{d}\chi = \mathrm{d}H\,. \tag{1.92}$$

If in addition the Killing vector field ∂_ψ of the Gibbons–Hawking base space turns out to be a Killing vector field of the full metric (1.90), the solution further simplifies, as in that case the 1-form ω in (1.90) can be written as

$$\omega = \omega_\psi(\mathrm{d}\psi + \chi) + \hat{\omega}\,, \tag{1.93}$$

with $\hat{\omega}$ and ω_ψ a 1-form and function on $\mathbb{R}^3$ respectively, and the full metric is found to be completely determined by four harmonic functions, H, K, L and M on $\mathbb{R}^3$, such that

$$f^{-1} = K^2H^{-1} + L\,, \tag{1.94}$$

$$\omega_\psi = H^{-2}K^3 + \frac{3}{2}H^{-1}KL + M\,, \tag{1.95}$$

and $\hat{\omega}$ satisfies

$$\star_3\,\mathrm{d}\hat{\omega} = H\,\mathrm{d}M - M\,\mathrm{d}H + \frac{3}{2}(K\,\mathrm{d}L - L\,\mathrm{d}K)\,. \tag{1.96}$$

The Maxwell field takes the form

$$F = \frac{\sqrt{3}}{2}\,\mathrm{d}\left[f(\mathrm{d}t + \omega) - \frac{K}{H}(\mathrm{d}\psi + \chi) - \xi\right]\,, \tag{1.97}$$

with a 1-form ξ on $\mathbb{R}^3$ satisfying

$$\star_3\,\mathrm{d}\xi = -\,\mathrm{d}K\,. \tag{1.98}$$

Minimal supergravity coupled to $N-1$ abelian gauge fields

The action (1.83) can be generalised to include multiple Maxwell fields. For $N-1$ additional abelian gauge fields (i.e. N Maxwell fields in total) this leads to [60] (for a summary see e.g. [51])

$$S_{\mathrm{bos}} = \frac{1}{2\kappa_5^2}\int\left(\star R - Q_{IJ}F^I\wedge\star F^J - Q_{IJ}\,\mathrm{d}X^I\wedge\star\,\mathrm{d}X^J - \frac{1}{6}C_{IJK}F^I\wedge F^J\wedge A^K\right)\,, \tag{1.99}$$

where the index $I = 1, \ldots, N$, $F^I = \mathrm{d}A^I$, the C_{IJK} are constants (usually taken to be symmetric in IJK) and the X^I are scalar fields satisfying

$$\frac{1}{6}C_{IJK}X^I X^J X^K = 1\,. \tag{1.100}$$

One may further express Q_{IJ} in terms of the scalar fields as

$$Q_{IJ} = \frac{9}{2}X_I X_J - \frac{1}{2}C_{IJK}X^K\,, \tag{1.101}$$

where we have defined

$$X_I = \frac{1}{6}C_{IJK}X^J X^K\,. \tag{1.102}$$

The equations of motion for the action (1.99) can be found for instance in [61]. As for the minimal theory, supersymmetric solutions again fall into a null and a timelike class. In the timelike class the metric is given by (1.90) and one can again choose a Gibbons–Hawking solution base space (1.91), (1.92). One then finds that the solution is determined by a set of $2N+2$ harmonic functions, H, K^I, L_I, M, such that [51]

$$f^{-1}X_I = \frac{1}{24}H^{-1}C_{IJK}K^J K^K + L_I\,, \tag{1.103}$$

$$\omega_\psi = -\frac{1}{48}H^{-2}C_{IJK}K^I K^J K^K - \frac{3}{4}H^{-1}L_I K^I + M\,, \tag{1.104}$$

and $\hat{\omega}$ satisfies

$$\star_3\, \mathrm{d}\hat{\omega} = H\,\mathrm{d}M - M\,\mathrm{d}H + \frac{3}{4}(L_I\,\mathrm{d}K^I - K^I\,\mathrm{d}L_I)\,. \tag{1.105}$$

The Maxwell fields are

$$F^I = \mathrm{d}\left[X^I f(\mathrm{d}t + \omega) + \frac{1}{2}\frac{K^I}{H}(\mathrm{d}\psi + \chi) + \frac{1}{2}\xi^I\right] \tag{1.106}$$

with

$$\star_3\, \mathrm{d}\xi^I = -\,\mathrm{d}K^I\,. \tag{1.107}$$

The minimal solution ($N = 1$) may be recovered by setting $C_{111} = \frac{2}{\sqrt{3}}$, $X^1 = \sqrt{3}$, $K^1 = -2\sqrt{3}K$, $L_1 = \frac{1}{\sqrt{3}}L$ and $F^1 = 2F$ [51].

It is worth noting that this theories may be derived as compactifications of eleven-dimensional supergravity on Calabi–Yau threefolds [20].

1.2.3 Supersymmetric black holes in five dimensions

Five-dimensional minimal supergravity provides a rich playground for higher-dimensional black hole solutions, which are discussed in this section.

BMPV revisited

One can easily see that the BMPV solution (1.81) is a solution to (1.84) by starting from a Gibbons–Hawking ansatz and choosing the harmonic functions

$$H = \frac{1}{r}\,, \qquad K = 0\,, \qquad L = 1 + \frac{\ell_0}{r}\,, \qquad M = \frac{m_0}{r}\,, \tag{1.108}$$

where we have written the $\mathbb{R}^3$-base of the Gibbons–Hawking space in spherical coordinates (r, θ, ϕ), so in particular r is the $\mathbb{R}^3$-radius. This gives the metric

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1+\frac{\ell_0}{r}\right)^{-2}\left[\mathrm{d}t + \frac{m_0}{r}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)\right]^2 \\ &+ \left(1+\frac{\ell_0}{r}\right)\left[r(\mathrm{d}\psi+\cos\theta\,\mathrm{d}\phi)^2 + \frac{1}{r}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2)\right]\,. \end{aligned} \tag{1.109}$$

with $0 \le \psi < 4\pi$, so (ψ, ϕ, θ) are Euler angles of the 3-sphere. Changing coordinates to the $\mathbb{R}^4$-radius $\rho^2 = 4r$, angles to $\phi_1 = \frac{1}{2}(\psi - \phi)$, $\phi_2 = \frac{1}{2}(\psi + \phi)$, $\theta \to 2\theta$, identifying $\mu = 4\ell_0$, $a = 8m_0$, one can see that this is indeed isometric to the BMPV solution (1.81). Similarly, the generalised 3-charge BMPV solution (1.75) of [116] may be obtained as a solution of $U(1)^3$-supergravity [24].

Supersymmetric black rings

Supersymmetric black holes with horizon topology $S^1 \times S^2$ were first discussed in [37]. These were found as solutions to five-dimensional minimal supergravity. The asymptotically flat, supersymmetric single black ring solution can be written in the Gibbons–Hawking framework with harmonic functions (see e.g. [51])

$$H = \frac{1}{r_1}\,, \qquad K = \frac{k_0}{r}\,, \qquad L = 1 + \frac{\ell_0}{r}\,, \qquad M = -\frac{3}{2}k_0 + \frac{3k_0 a_1}{2r}\,, \tag{1.110}$$

where H has a centre at $r_1 \equiv \sqrt{r^2 + a_1^2 - 2a_1 r\cos\theta} = 0$. The resulting metric is smooth at $r_1 = 0$ and has a horizon of topology $S^1 \times S^2$ at $r = 0$, [14]

$$\mathrm{d}s^2_{\mathcal{H}} = \frac{3\ell_0^2 - 12k_0^2 a_1}{k_0^2}\,\mathrm{d}\psi^2 + k_0^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{1.111}$$

such that the horizon has area $A_{\mathcal{H}} = 8\pi^2\sqrt{3}|k_0|(\ell_0^2 - 4k_0^2 a_1)^{1/2}$. A single supersymmetric black ring solution has two non-vanishing angular momenta with respect to the two orthogonal rotation planes at infinity. Moreover these angular momenta are necessarily non-equal. This in particular implies that it will never have the same conserved charges as the BMPV solution (for which the two angular momenta are equal).

The supersymmetric black ring solution can be generalised to multiple (concentric) black rings [50] or further to a (multi-black ring) solution of $U(1)^N$-supergravity [51].

[14] Note that this is slighty different to the expression in [51] as we have kept ψ at period 4π.

Interestingly, other than the single black rings, these solutions allow for equal angular momenta, so their asymptotic charges may overlap with those of the BMPV black hole. This provides yet another example of black hole non-uniqueness in higher dimensions. Moreover, within the class of equal-angular-momenta multi-black rings, it is possible to find both, solutions with entropy exceeding that of the BMPV solution, or such with lower entropy [50].

Black lenses

We have seen in section 1.1.1 that the higher-dimensional horizon topology theorem allows for horizons with lens space topology. Such solutions have first been constructed in [86]. They may again be written as a Gibbons–Hawking solution of minimal supergravity with harmonic functions [15]

$$\begin{aligned} H &= \frac{2}{r} - \frac{1}{r_1}\,, & L &= 1 + \frac{\ell_0}{r} + \frac{k_1^2}{r_1} \\ K &= \frac{k_1}{r_1}\,, & M &= -\frac{3k_1}{2} + \frac{k_1(3\ell_0 - 2k_1^2 + 6a_1)}{2r} + \frac{k_1^3}{2r_1}\,, \end{aligned} \tag{1.112}$$

The corresponding solution is asymptotically flat, and is smooth on and outside a horizon at $r = 0$. From the near horizon analysis one finds that cross-sections of the horizon have metric [88]

$$\mathrm{d}s^2_{\mathcal{H}} = \left[\frac{\ell_0}{2} - \frac{k_1^2}{4\ell_0^2}(3\ell_0 - 2k_1 + 6a_1)^2\right](\mathrm{d}\psi + 2\cos\theta\,\mathrm{d}\phi)^2 + 2\ell_0(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,. \tag{1.113}$$

While this looks locally like the BMPV solution, this is not true globally: Since the periods of all angular variables are fixed, this is the metric of a quotient $S^3/\mathbb{Z}_2 \cong L(2,1)$. The area of the horizon can be computed from (1.113) and is given by

$$A_{\mathcal{H}} = 16\pi^2\sqrt{2\ell_0^3 - k_1^2(3\ell_0 - 2k_1 + 6a_1)^2}\,. \tag{1.114}$$

Much like the single black ring, it was shown in [86] that the black $L(2,1)$ solution does not allow for equal angular momenta (with respect to the orthogonal rotation planes at infinity). This means in particular that the black lens cannot carry the same asymptotic charges as the BMPV black hole.

The above construction has been generalised in [112] to construct black holes of horizon topology $L(p,1)$ for general $p \geq 2$. Again, these solutions may not have equal angular momenta. Further generalisations to solutions of $U(1)^3$-supergravity can be found in [88, 110]. Yet again, these lenses were found to not overlap with the BMPV in any region of their parameter space.

It is worth noting that so far, together with the solutions presented in chapter 3, these are the only known solutions with horizons of lens space topology. In particular

[15] Note that we use a different gauge as in [86].

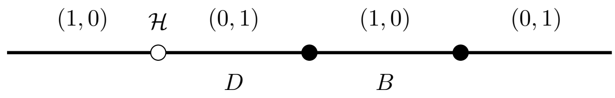


Figure 1.5: Rod diagram of the spherical black hole with nontrivial topology solution. There are two nontrivial two-cycles: A noncontractible *bubble* B and a noncontractible disc D attached to the horizon. The numbers above each axis rod specify the biaxial Killing vector that vanishes on the rod with respect to the 2π-periodic basis (v_-, v_+), $v_\pm = \partial_\phi \mp \partial_\psi$.

no non-supersymmetric version of these solutions (i.e. which are asymptotically flat, everywhere regular on an outside a horizon, and without conical singularities or closed timelike curves) have been constructed so far. Recent progress in this matter has been made in [111] using the inverse scattering method [10], however despite the solution being free of conical or curvature singularities, the issue of naked CTCs remains.

Black holes with nontrivial spacetime topology

Topological censorship [46] implies that the DOC of an asymptotically flat and globally hyperbolic spacetime must be simply connected. In four dimensions this implies that Cauchy surfaces must be topologically trivial. For higher-dimensional solutions however, simple connectedness allows for nontrivial topological structures outside the black hole horizon. Indeed, such a black hole with nontrivial spacetime topology was constructed in [85]. This is again a solution to five-dimensional minimal supergravity with a Gibbons–Hawking ansatz and harmonic functions

$$\begin{aligned} H &= \frac{1}{r} - \frac{1}{r_1} + \frac{1}{r_2}\,, & L &= 1 + \frac{\ell_0}{r} + \frac{k_1^2}{r_1} - \frac{k_2^2}{r_2} \\ K &= \frac{k_1}{r_1} + \frac{k_2}{r_2}\,, & M &= -\frac{3}{2}(k_1 + k_2) + \frac{m_0}{r} + \frac{k_1^3}{2r_1} + \frac{k_2^3}{2r_2}\,, \end{aligned} \tag{1.115}$$

where $r_{1,2} = \sqrt{r^2 + a_{1,2}^2 - 2ra_{1,2}\cos\theta}$ and m_0 and ℓ_0 may be expressed in terms of k_1 and k_2. This defines an asymptotically flat solution with a spherical horizon at $r = 0$ with area $A_{\mathcal{H}} = 16\pi^2\sqrt{\ell_0^3 - m_0^2}$. The solution is smooth at the other two centres, $r_1 = 0$, $r_2 = 0$. Let us consider the rod structure of this solution, which is depicted in Figure 1.5. Note that since we are dealing with extremal solutions, the horizon interval (labelled $\mathcal{H}$) is collapsed to a single point on the axis. This corresponds to the point $r = 0$. The two filled black points on the axis correspond to the points $r_1 = 0$ and $r_2 = 0$ respectively. At these two points both Killing fields vanish. The interval between $r_1 = 0$ and $r_2 = 0$ thus corresponds to a noncontractible *bubble* (with endpoints corresponding to poles). This solution has therefore often been referred to as *black hole with bubble*. The rod structure shows that there is another nontrivial topology contained in the spacetime, corresponding to the interval between the horizon and the first centre. Again both biaxial Killing fields vanish at one of the endpoints

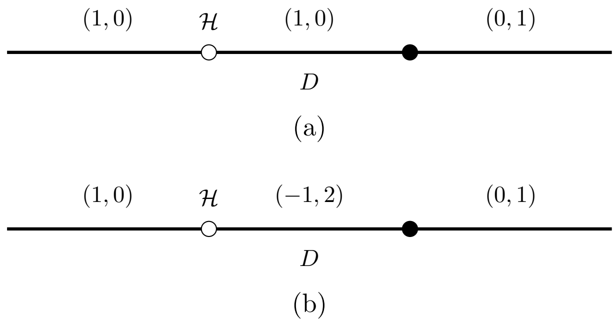


Figure 1.6: Rod structures for (a) the supersymmetric black ring and (b) the black $L(2, 1)$ lens. Both have a disc topology in their DOC.

(the one corresponding to $r_1 = 0$), but this is not the case on the horizon, such that the interval corresponds to a disc centred at $r_1 = 0$ and attached to the horizon.

In fact, both the black ring solution, as well as the black lens solution discussed above also contain nontrivial topologies in the DOC. Their rod structures are depicted in Figure 1.6. Both, the single black ring (1.110), Figure 1.6(a) and the black $L(2, 1)$, Figure 1.6(b), have a noncontractible disc topology centred at $r_1 = 0$ and attached to the horizon.

Finally, the black hole with nontrivial topology solutions provide a further example of black hole non-uniqueness. As was shown in [85], these solutions allow for equal angular momenta, hence their asymptotic charges overlap with those of BMPV in some region of the phase space. This region was analysed in detail in [79]. In particular, in the region of overlap there exists a subregion where the black hole with nontrivial topology has entropy exceeding that of the BMPV black hole. This is discussed in length in a more general context in chapter 4.

Chapter 2

A classification of supersymmetric black holes in five dimensions

2.1 Introduction

The classification of isolated gravitating equilibrium states is a problem of central importance in general relativity. As we have seen in chapter 1, this is sufficiently well understood in four dimensions: Firstly, by the horizon topology theorem, connected components of cross-sections of the horizon of a stationary, asymptotically flat black hole solution in $D = 4$ must be of spherical topology. Restricting to four-dimensional Einstein–Maxwell theory (with certain assumptions), by the no-soliton theorem any nontrivial, stationary and asymptotically flat solution must contain a black hole (see e.g. [26]), and by the four-dimensional uniqueness theorem, Theorem 1.3, any black hole solution must be a Kerr–Newman solution. Thus solutions are completely classified by the phase space of Kerr–Newman solutions.

We have also seen, however, that the problem is much more complex in dimensions greater than four. The construction of the black ring solution has shown not only that there do exist black hole solutions with non-spherical horizon topologies, but also that asymptotic charges are not sufficient to uniquely determine solutions in higher dimensions. This also applies to the simpler (because more restricted) class of supersymmetric solutions, where the family of supersymmetric black ring solutions exhibits non-uniqueness within itself [38]. Moreover, the so-called *bubbling* spacetimes in supergravity constructed in [12], which are asymptotically flat and everywhere smooth solutions, yet are topologically nontrivial as they contain noncontractible 2-cycles (*bubbles*), have provided examples of soliton solutions in higher dimensions. In fact, these two examples show that all the above features complicating the space of higher dimensional black hole solutions—non-spherical black holes, black hole non-uniqueness, and existence of solitons—are already present for supersymmetric solutions in five dimensions.

Notwithstanding the complications arising when leaving the realm of four dimensions, a number of results have been derived that help constrain the topology and symmetry of higher-dimensional, asymptotically flat black hole spacetimes. Even though some of these results can be generalised to arbitrary dimension, we will from here on focus on five-dimensional solutions only. First, as in four dimensions, topological censorship implies the domain of outer communication must be simply connected [46]. The generalised horizon topology theorem of Galloway and Schoen [47] (see section 1.1.1) restricts horizon topologies to one of S^3, $S^1 \times S^2$, S^3/Γ (with Γ a finite subgroup of $SO(4)$) or connected sums thereof [72]. Further, by the rigidity theorem, any analytic, stationary and rotating black hole solution must also be axisymmetric, i.e. possess an isometry group $U(1) \times \mathbb{R}$ [74].

On top of these global results, as was discussed in more detail in section 1.2.2 of the preliminaries, for five-dimensional minimal supergravity, supersymmetric solutions have been classified locally [53]. It is a natural question to ask, whether one can combine these known local constraints with the known global constraints to yield a full classification of black hole solutions in $D = 5$ minimal supergravity.

It is interesting to note that while the rigidity theorem only guarantees the existence of an $\mathbb{R} \times U(1)$-symmetry for stationary, rotating solutions, in fact all known solutions possess $\mathbb{R} \times U(1)^2$-symmetry (stationary and "biaxisymmetric"). Hence they fall into the class of generalised Weyl solutions [42, 63], and one can assign a rod structure (see Definition 1.2) to them. Yet this still leaves one important question unanswered: For what rod structures and asymptotic charges do solutions exist? Therefore even for supersymmetric and biaxisymmetric solutions, a full classification is unknown.

Chapters 2 and 3 of this thesis give such a classification of supersymmetric and biaxisymmetric solutions in minimal supergravity in five dimensions. One of the main results is the following classification theorem, the complete statement of which is given in Theorem 2.5.

Theorem 2.1. *Consider an asymptotically flat, supersymmetric and biaxisymmetric solution to five-dimensional minimal supergravity with a globally hyperbolic domain of outer communication, possibly containing a black hole. Then, the solution must have a Gibbons–Hawking base and the associated harmonic functions are of multi-centred type.*

The general structure of this chapter is as follows. First, general Gibbons–Hawking solutions to minimal supergravity are studied in section 2.2. In section 2.3 we derive the important result that any asymptotically flat, supersymmetric and biaxisymmetric solution must in fact fall into the class of Gibbons–Hawking solutions. Moreover, we find that smoothness of the solution is linked to the behaviour of the harmonic functions at specific points in the spacetime: the event horizon (in the case of a black hole solution), and fixed points of the imposed $U(1)^2$-symmetry. Sections 2.4 and 2.5 are dedicated to these special points. Arguments presented in these sections typically involve combining local constraints stemming from supersymmetry with global constraints from asymptotic flatness. The final result is given in section 2.6.

The results presented in this chapter have first been published in [16].

2.2 Supersymmetric solutions with Gibbons–Hawking base

As we have seen in the section 1.2.2, the bosonic sector of five-dimensional minimal supergravity is Einstein–Maxwell theory coupled to a Chern–Simons term, and it was shown in [53] that, given a Killing spinor, one can construct a smooth function f and a Killing vector V, each quadratic in the spinor, such that $V \cdot V = -f^2$. Thus V must be either null or timelike (at least in some region). Solutions where V is null can be fully determined and correspond to plane wave and pp-wave spacetimes. We will be interested in asymptotically flat solutions, possibly containing a black hole, which must be in the timelike class.

In any region where the supersymmetric Killing field V is timelike, the spacetime metric takes the general form (1.90). Solutions significantly simplify if the base space in (1.90) is a Gibbons–Hawking space. In fact, as we explain later, under the additional assumption of biaxial symmetry, the base space *must* be of this simpler form. We will study the class of Gibbons–Hawking solutions in further detail here. It is worth stressing that our analysis in this section does *not* assume biaxial symmetry, hence is valid for any supersymmetric solution with a Gibbons–Hawking base.

2.2.1 Spacetime invariants

We wish to perform a global analysis of this family of local metrics. To this end, it will be useful to record the spacetime invariants

$$\begin{aligned} V \cdot V = g_{tt} &= -f^2 = -\frac{H^2}{(K^2+HL)^2}\,, \\ \partial_\psi \cdot \partial_\psi = g_{\psi\psi} &= \frac{1}{fH} - f^2\omega_\psi^2 = -\frac{4H^2M^2 + 12HKLM - 4HL^3 + 8K^3M - 3K^2L^2}{4(HL+K^2)^2}\,, \\ V \cdot \partial_\psi = g_{t\psi} &= -f^2\omega_\psi = -\frac{H^2M + \frac{3}{2}HKL + K^3}{(K^2+HL)^2}\,, \end{aligned} \tag{2.1}$$

and

$$\iota_V F = -\frac{\sqrt{3}}{2}\,\mathrm{d}f\,, \qquad \iota_{\partial_\psi} F = -\frac{\sqrt{3}}{2}\,\mathrm{d}\left(f\omega_\psi - \frac{K}{H}\right) = -\frac{\sqrt{3}}{2}\,\mathrm{d}\left(\frac{HM + \frac{1}{2}KL}{HL+K^2}\right). \tag{2.2}$$

Note that since $\mathrm{d}\iota_V F = \mathrm{d}\iota_{\partial_\psi} F = 0$, for a simply connected spacetime (as we will be interested in) one can deduce the existence of two globally defined functions Φ, Ψ satisfying

$$\iota_V F = \frac{\sqrt{3}}{2}\,\mathrm{d}\Phi, \qquad \iota_{\partial_\psi} F = \frac{\sqrt{3}}{2}\,\mathrm{d}\Psi\,. \tag{2.3}$$

These functions Φ, Ψ are the electric potential and a magnetic potential respectively. From (1.97) we can identify these potentials up to an additive constant as

$$\Phi = -f, \qquad \Psi = -f\omega_\psi + KH^{-1}\,, \tag{2.4}$$

establishing that the right-hand sides of (2.4) are indeed globally defined, smooth functions on the spacetime (note the former is true even without a Gibbons–Hawking base). Up to a choice of gauge, these can be identified with components A_t, A_ψ of the gauge field, (1.97). In fact, smoothness of f also follows more easily from the fact that f is itself a spacetime invariant, as it can be constructed as a bilinear in the Killing spinor, see (1.86).

For later convenience we can rewrite the solution as

$$\begin{aligned} \mathrm{d}s^2 &= g_{tt}(\mathrm{d}t+\hat\omega)^2 + 2g_{t\psi}(\mathrm{d}t+\hat\omega)(\mathrm{d}\psi+\chi) + g_{\psi\psi}(\mathrm{d}\psi+\chi)^2 + \frac{H}{f}\,\mathrm{d}x^i\,\mathrm{d}x^i\,, \\ A &= A_t(\mathrm{d}t+\hat\omega) + A_\psi(\mathrm{d}\psi+\chi) - \xi\,. \end{aligned} \tag{2.5}$$

The inverse metric can be written as

$$\begin{aligned} g^{tt} &= -\frac{H}{f}g_{\psi\psi} + \frac{f}{H}\hat\omega_i\hat\omega_i\,, & g^{ti} &= -\frac{f}{H}\hat\omega_i\,, \\ g^{t\psi} &= \frac{H}{f}g_{t\psi} + \frac{f}{H}\hat\omega_i\chi_i\,, & g^{\psi i} &= -\frac{f}{H}\chi_i\,, \\ g^{\psi\psi} &= -\frac{H}{f}g_{tt} + \frac{f}{H}\chi_i\chi_i\,, & g^{ij} &= \frac{f}{H}\delta_{ij}\,, \end{aligned} \tag{2.6}$$

and the determinant of the metric is

$$\sqrt{-\det g} = \frac{H}{f} = K^2 + HL\,. \tag{2.7}$$

We now provide a coordinate-independent interpretation of the harmonic functions. To do so, let us first define the determinant of the matrix of inner products of the Killing fields ∂_t and ∂_ψ,

$$N \equiv -\begin{vmatrix} g_{tt} & g_{t\psi} \\ g_{t\psi} & g_{\psi\psi} \end{vmatrix}\,. \tag{2.8}$$

This will be a key invariant in our analysis. From (2.1) it follows that $N = f/H$, so we can write the harmonic function H as

$$H = \frac{f}{N}\,. \tag{2.9}$$

Next, we relate the harmonic functions K, L, M to invariants as follows. From (2.2) one sees that

$$KH^{-1} = \Psi - \frac{g_{t\psi}}{f} \tag{2.10}$$

and hence

$$K = \frac{1}{N} \left(f\Psi - g_{t\psi} \right) . \tag{2.11}$$

Using (1.94) and (1.95), together with the above expression for K gives, after a little algebra,

$$L = \frac{1}{N} \left(f g_{\psi\psi} + 2 g_{t\psi} \Psi - f\Psi^2 \right) , \tag{2.12}$$

$$M = \frac{1}{2N} \left(g_{\psi\psi} g_{t\psi} - 3 f \Psi g_{\psi\psi} - 3\Psi^2 g_{t\psi} + f\Psi^3 \right) . \tag{2.13}$$

Observe that K is only defined up to

$$K \to K + cH \,, \tag{2.14}$$

where c is a constant corresponding to the integration constant for Ψ. It follows that L, M are defined up to the shifts

$$L \to L - 2cK - c^2 H \,, \qquad M \to M - \tfrac{3}{2} cL + \tfrac{3}{2} c^2 K + \tfrac{1}{2} c^3 H \,. \tag{2.15}$$

The following result will be useful for a global analysis of solutions with a Gibbons–Hawking base.

Lemma 2.1. *Let (g, F) be a supersymmetric solution to minimal supergravity with a Gibbons–Hawking base, H, K, L, M the associated harmonic functions defined up to* (2.14, 2.15), and let N be defined as in (2.8).

1. If H, K, L, M are smooth and

$$K^2 + HL > 0 \,, \tag{2.16}$$

   then (g, F) is smooth, g^{-1} exists and is smooth, and $N > 0$.

2. Conversely, if (g, F) is smooth and $N > 0$, then H, K, L, M are smooth and obey (2.16).

Proof. Smoothness of H, K, L, M implies that the 1-forms $\chi, \hat{\omega}, \xi$ must be smooth (otherwise their exterior derivatives (1.92, 1.96, 1.98) would not be). Then, from (2.1, 2.2), (2.5, 2.6), and (2.16) it follows that: (i) all components of the metric are smooth, (ii) a smooth inverse metric exists, and (iii) the Maxwell field is smooth. Furthemore, equation (2.9) shows that

$$N^{-1} = K^2 + HL \,, \tag{2.17}$$

so (2.16) is equivalent to $N > 0$. Therefore we have established part 1 of the Lemma. Part 2 immediately follows from (2.9, 2.11–2.13). □

Remarks

1. As we show in the next section, in the context of asymptotically flat, supersymmetric and biaxisymmetric spacetimes with a globally hyperbolic domain of outer communication, the invariant $N \geq 0$ on and outside any black hole region and vanishes only in two instances: (i) at fixed points of the triholomorphic Killing field $\partial_\psi = 0$, or (ii) on an event horizon. We will analyse (i) and (ii) later making more detailed use of the biaxial symmetry together with certain global constraints.

2. We will require the spacetime to be stably causal. In particular we will require t to be a time function, so
$$g^{tt} < 0\,, \tag{2.18}$$
which implies the absence of CTCs. In particular, as can be seen from (2.6), $g^{tt} < 0$ and $N > 0$ imply that $g_{\psi\psi} > 0$.

3. A priori, the metric in the chart (t, ψ, x^i) is only defined in a region where $f \neq 0$, i.e. where V is timelike. However, Lemma 2.1 shows, that if $N > 0$, g is smooth with smooth harmonic functions, even if $f = 0$. In fact, if f vanishes on a smooth hypersurface[1], this hypersurface must be a an *evanescent ergosurface* [56], that is, a smooth, timelike hypersurface on which the elsewhere timelike Killing field becomes null. This can be seen as follows. First, since $N = f/H$, in a region where $N > 0$ one finds that $f = 0$ if and only if $H = 0$. Since we take $f = 0$ to be a smooth hypersurface, $\mathrm{d}f \neq 0$ on the hypersurface and thus also $\mathrm{d}H|_{f=0} = \frac{1}{N}\,\mathrm{d}f|_{f=0} \neq 0$. In particular this means that near $f = 0$ we may introduce coordinates (H, y^A) on $\mathbb{R}^3$ so that
$$\mathrm{d}x^i\,\mathrm{d}x^i = \varrho^2\,\mathrm{d}H^2 + \tilde{g}_{AB}\,\mathrm{d}y^A\,\mathrm{d}y^B\,. \tag{2.19}$$
Then harmonicity of H implies that
$$0 = \mathrm{d} \star_3 \mathrm{d}H \sim \partial_H \left(\left(\frac{\det \tilde{g}}{\rho^2} \right)^{1/2} \right) \mathrm{d}H \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2\,, \tag{2.20}$$
so $\det \tilde{g} = \rho^2 F(y)^2$ for some function $F(y)$. We may always choose the coordinates y^A such that on $H = 0$ we have $\det \tilde{g} = \rho^2$ and hence $\mathrm{d}\chi = \star_3\,\mathrm{d}H = \rho^{-1} \star_2 1$ where $\star_2$ is the Hodge dual on the 2-dimensional space with metric $\tilde{g}$. With this,

[1] It is possible that the zero set of f is not always a hypersurface; we will not analyse this possibility here. In any case, our analysis will not assume this.

on the hypersurface $f = 0$ we may write the solution (2.5) as

$$
\begin{aligned}
\mathrm{d}s^2|_{H=0} &= -\frac{2}{K}(\mathrm{d}t + \hat{\omega}_A\,\mathrm{d}y^A)(\mathrm{d}\psi + \tilde{\chi}_A\,\mathrm{d}y^A) \\
&\quad + \frac{3L^2 - 8KM}{4K^2}(\mathrm{d}\psi + \tilde{\chi}_A\,\mathrm{d}y^A)^2 + K^2\tilde{g}_{AB}\,\mathrm{d}y^A\,\mathrm{d}y^B\,, \\
A|_{H=0} &= \frac{\sqrt{3}}{2}\Big(\frac{L}{2K}(\mathrm{d}\psi + \tilde{\chi}_A\,\mathrm{d}y^A) - \xi_A\,\mathrm{d}y^A\Big)\,.
\end{aligned}
\tag{2.21}
$$

We know that all the harmonic functions (and hence also the one-forms χ, $\hat{\omega}$, ξ) are smooth on $f = 0$. Further from $N > 0$ it follows that on $H = 0$, $K \neq 0$, so $f = 0$ indeed corresponds to a smooth timelike hypersurface, hence an evanescent ergosurface. In fact, it has been shown [95] that any supersymmetric solution to minimal supergravity is smooth at an evanescent ergosurface if and only if the hyper-Kähler base is ambipolar (according to their definition) and ω has a particular behaviour near the ergosurface.

2.2.2 Asymptotic flatness

We will be interested in asymptotically flat (i.e. Minkowskian) solutions, which in particular requires ψ to be rotational. For orientation, let us write Minkowski space in the Gibbons–Hawking framework. First, vanishing of the Maxwell field in the timelike class implies that f is a constant and ω is pure gauge, $\mathrm{d}\omega = 0$. Without loss of generality we may always set $\omega = 0$ and $f = 1$. Then the Gibbons–Hawking base space must be Euclidean space $\mathbb{R}^4$. For ψ rotational this implies that without loss of generality one can set

$$
H = \frac{1}{r}\,, \qquad L = 1\,, \qquad K = M = 0\,, \qquad \chi = \cos\theta\,\mathrm{d}\phi\,, \qquad \hat{\omega} = \xi = 0\,, \tag{2.22}
$$

where we have written the $\mathbb{R}^3$ base in spherical coordinates (r, θ, ϕ). Upon the coordinate change $r = \rho^2/4$ the metric is

$$
\mathrm{d}s^2_{\text{Mink}} = -\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \tfrac{1}{4}\rho^2[(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2]\,, \tag{2.23}
$$

where now the spatial $\mathbb{R}^4$ is in polar coordinates with the round S^3 written in terms of Euler angles (θ, ϕ, ψ). The Euler angles ϕ and ψ are related to independently 2π-periodic angles $\phi^\pm$ in orthogonal planes via

$$
\phi^\pm = \tfrac{1}{2}(\phi \mp \psi)\,, \qquad v_\pm \equiv \partial_{\phi^\pm} = \partial_\phi \mp \partial_\psi\,. \tag{2.24}
$$

Note $v_+ = 0$ on $\theta = 0$ and $v_- = 0$ on $\theta = \pi$ represent the two axes which extend out to infinity. In terms of Euler angles the periodicities are generated by the identifications $(\psi, \phi) \sim (\psi + 4\pi, \phi)$ and $(\psi, \phi) \sim (\psi + 2\pi, \phi + 2\pi)$.

The asymptotic expansion of an asymptotically flat spacetime is particularly simple for this class of metrics. Requiring that the harmonic functions H, K, L, M asymptotically approach those of Minkowski spacetime (up to the freedom (2.14, 2.15)) implies they can be written as a standard multipole expansion

$$\begin{aligned} H &= \frac{1}{r} + \sum_{l\geq 1,m} h_{lm} Y_{lm}(\theta,\phi) r^{-l-1}\,, \\ K &= \frac{k_\infty}{r} + \sum_{l\geq 1,m} k_{lm} Y_{lm}(\theta,\phi) r^{-l-1}\,, \\ L &= 1 + \frac{\ell_\infty}{r} + \sum_{l\geq 1,m} \ell_{lm} Y_{lm}(\theta,\phi) r^{-l-1}\,, \\ M &= m + \frac{m_\infty}{r} + \sum_{l\geq 1,m} m_{lm} Y_{lm}(\theta,\phi) r^{-l-1}\,, \end{aligned} \tag{2.25}$$

where $k_\infty, \ell_\infty, m_\infty, m$ are constants. The constant k_∞ is pure gauge and can be set to any value using (2.14). We have included a constant m in M in order not to fix the corresponding gauge freedom (2.15). The above form for the harmonic functions then determines the asymptotic expansion of the spacetime. In particular, this implies $f = 1 + \mathcal{O}(r^{-1})$ so the supersymmetric Killing vector V is timelike near infinity, i.e. V is the stationary Killing field. Furthermore, $\omega_\psi = m + \frac{3}{2}k_\infty + \mathcal{O}(r^{-1})$ so we set

$$m = -\tfrac{3}{2}k_\infty\,, \tag{2.26}$$

which is indeed invariant under the gauge transformations (2.14) and (2.15). When integrating for χ and $\hat{\omega}$ we will also set the integration constants so that $\chi = \cos\theta\,\mathrm{d}\phi + \mathcal{O}(r^{-1})$ and $\hat{\omega} = \mathcal{O}(r^{-1})$. Without loss of generality, we may always make such choices for asymptotically flat solutions. It is now clear that the smoothness condition (2.16) and causality condition (2.18) are satisfied in the asymptotic region.

2.3 Supersymmetric and biaxisymmetric spacetimes

We now want to study supersymmetric solutions $(\mathcal{M}, g, F)$ in the timelike class that in addition have biaxial symmetry. One of our main results in this section is that this implies the solution must be of Gibbons–Hawking type, which was studied in detail in the previous section.

To be precise, we make the following assumptions on $(\mathcal{M}, g, F)$.

Definition 2.1. *Let $(\mathcal{M}, g, F)$ be a supersymmetric solution to five-dimensional minimal supergravity in the timelike class. We call the solution* biaxisymmetric *if*

(i) there is a $U(1)^2$-isometry, generated by Killing fields m_i, $i = 1, 2$, whose orbits are 2π periodic;

(ii) $[V, m_i] = 0$ where V is the supersymmetric Killing field; and

(iii) $\mathcal{L}_{m_i} F = 0$.

Clearly, the biaxial Killing fields are defined up to a transformation $m_i \to A_{ij} m_j$ where A_{ij} is an $SL(2,\mathbb{Z})$ matrix. We will sometimes denote the Killing fields collectively by K_A, where $A = 0, 1, 2$ and $K_0 = V$ and $K_i = m_i$.

The above conditions mean the spacetime is stationary, where the supersymmetric Killing field V is the stationary Killing field, and in addition has two axial symmetries which commute with V. Therefore our solutions may be seen as supersymmetric versions of the stationary and biaxisymmetric vacuum solutions studied in [63, 76]. The additional assumption of supersymmetry places extra local and global constraints on the solution which shall be explored below.

Let us first study the restrictions put on the local form of timelike solution $(\mathcal{M}, g, F)$ by the additional assumptions of asymptotic flatness and biaxisymmetry. Our results are summarised in the following lemma.

Lemma 2.2. *Consider an asymptotically flat, supersymmetric and biaxisymmetric solution to minimal supergravity with supersymmetric Killing field $V = \partial_t$. Then the hyper-Kähler base must be a Gibbons–Hawking metric (1.91) whose triholomorphic Killing field ∂_ψ leaves the full solution invariant. Furthermore, the harmonic functions H, K, L, M on $\mathbb{R}^3$ are axisymmetric and the 1-forms can be written as*[2]

$$\chi = \chi(\rho, z)\, \mathrm{d}\phi\,, \qquad \hat{\omega} = \hat{\omega}(\rho, z)\, \mathrm{d}\phi\,, \qquad \xi = \xi(\rho, z)\, \mathrm{d}\phi\,, \tag{2.27}$$

where (ρ, z, ϕ) are cylindrical coordinates on $\mathbb{R}^3$. In particular, in the coordinates $y^A = (t, \psi, \phi)$ and $z^a = (\rho, z)$, the spacetime metric (1.90) then takes the block diagonal form

$$\mathrm{d}s^2 = G_{AB}(z^a)\, \mathrm{d}y^A\, \mathrm{d}y^B + q_{ab}(z^a)\, \mathrm{d}z^a\, \mathrm{d}z^b\,, \tag{2.28}$$

where

$$G_{AB}\, \mathrm{d}y^A\, \mathrm{d}y^B = -f^2(\mathrm{d}t + \omega_\psi(\mathrm{d}\psi + \chi\, \mathrm{d}\phi) + \hat{\omega}\, \mathrm{d}\phi)^2 + f^{-1}H^{-1}(\mathrm{d}\psi + \chi\, \mathrm{d}\phi)^2 + \frac{H\rho^2}{f}\, \mathrm{d}\phi^2 \tag{2.29}$$

is the inner product on the space spanned by the Killing fields $\{\partial_t, \partial_\psi, \partial_\phi\}$, and

$$q_{ab}\, \mathrm{d}z^a\, \mathrm{d}z^b = \frac{H}{f}(\mathrm{d}\rho^2 + \mathrm{d}z^2) \tag{2.30}$$

is a metric on surfaces orthogonal to the space of Killing fields. The determinant of G_{AB} is

$$\det G_{AB} = -\rho^2\,, \tag{2.31}$$

so (ρ, z) are in fact standard Weyl coordinates.

[2] For notational simplicity we denote both the 1-forms and their ϕ-components by χ, $\hat{\omega}$, ξ. Distinction between these will be apparent from context, or clarified if necessary.

Proof. To show the first part of the lemma, note that the biaxisymmetric Killing fields leave the base space metric h, as well as f and ω invariant. This can be seen as follows. First, since the m_i commute with the supersymmetric Killing field V, invariance of f follows from $f^2 = -V_\mu V^\mu$. Writing the base space metric as

$$h_{\mu\nu} = f g_{\mu\nu} + f^{-1} V_\mu V_\nu \,, \tag{2.32}$$

it follows that also $\mathcal{L}_{m_i} h = 0$. Lastly, we may always choose $\mathcal{L}_{m_i} t = 0$ by a suitable shift in the time function $t \to t + \lambda$ by a function λ on the base space, such that invariance of the one-form $V_\mu \, \mathrm{d}x^\mu = -f^2(\mathrm{d}t + \omega)$ implies that $\mathcal{L}_{m_i}\omega = -\mathcal{L}_{m_i} \, \mathrm{d}t = 0$. We may now apply the following result [55]: *A hyper-Kähler metric with a local $U(1)^2$-isometry is a Gibbons–Hawking metric and the triholomorphic Killing field is a linear combination of the $U(1)^2$-Killing fields.* Thus the base space has to be Gibbons–Hawking, with the triholomorphic Killing field ∂_ψ a linear combination of the m_i. Since both m_i leave the full solution invariant, so does ∂_ψ, completing the proof of the first part of the lemma.

Now in addition to the $U(1)$-isometry generated by the triholomorphic Killing field, the base space has an additional symmetry generated by a different (linearly independent) linear combination of the m_i, $\eta = \eta^i m_i$. Moreover, by an analogous argument to the above argument for five-dimensional spacetimes, η must in fact leave H, χ, as well as the $\mathbb{R}^3$ base invariant: Writing H as an invariant on the base space, $H^{-1} = h(\partial_\psi, \partial_\psi)$, shows that H is invariant under η. Furthermore invariance of $h(\partial_\psi, \cdot) = H^{-1}(\mathrm{d}\psi + \chi)$ implies invariance of the $\mathbb{R}^3$ base $\delta = H^{-1}h - h(\partial_\psi, \cdot)^2$, and we may always shift $\psi \to \psi + \tilde{\lambda}$ by a function $\tilde{\lambda}$ on $\mathbb{R}^3$ such that $\mathcal{L}_\eta \psi = 0$ and hence also $\mathcal{L}_\eta \chi = 0$. In particular this implies that η corresponds to a 1-parameter family of isometries of $\mathbb{R}^3$. As such it can be either a translation, a rotation, or a combination of the two (corkscrew). We will show that η has to correspond to a compact subgroup of the Euclidean group. Assume that η corresponds a non-compact group of isometries, i.e. a translation or corkscrew. Then its orbit curves are unbounded in $\mathbb{R}^3$. Since we know from asymptotic flatness that $H \to 0$ as the $\mathbb{R}^3$ radius $r \to \infty$, invariance of H under η would imply $H \equiv 0$, which is a contradiction. Hence η must correspond to a rotation, and we may write the $\mathbb{R}^3$ in cylindrical coordinates (ρ, ϕ, z) such that $\eta = \partial_\phi$ and in particular H does not depend on ϕ, $H = H(\rho, z)$.

We want to show that also the other harmonic functions, K, L and M are axisymmetric. Note from (2.11–2.13) that K, L and M are linked to the Maxwell field. By assumption, F is invariant under the three commuting Killing fields K_A, $\mathcal{L}_{K_A} F = 0$. Generalising standard arguments for stationary and axisymmetric solutions (see e.g. [106]), we recall that for Maxwell fields invariant under three commuting Killing fields, as a consequence of the Bianchi identity the functions $\iota_{K_A}\iota_{K_B} F$ are constant in the spacetime (see e.g. [87]). Furthermore, asymptotic flatness implies that a different linear combination of $K_i = m_i$, for $i = 1, 2$, vanish on the two axes which intersect the S^3 at infinity, namely v_+ and v_- in (2.24). Therefore all these constants must in fact vanish, $\iota_{K_A}\iota_{K_B}F = 0$. In particular, since the axial Killing field ∂_ϕ and the triholomorphic Killing field ∂_ψ must be linear combinations of the $K_i = m_i$ for $i = 1, 2$, we must have $\iota_{\partial_\phi}\iota_{\partial_\psi}F = 0$, so by (2.3) the magnetic potential is axisymmetric, $\mathcal{L}_{\partial_\phi}\Psi = 0$. Hence (2.4) implies that KH^{-1}—and therefore by axisymmetry of H, also K—must be axisymmetric. Axisymmetry of L and M then follows from invariance of f and ω_ψ under ∂_ϕ, together with (1.94) and (1.95). It remains to show that the one-forms χ, ω and ξ are gauge equivalent to (2.27). From $H = H(\rho, z)$ together with $\star_3\, \mathrm{d}\chi = \mathrm{d}H$ it follows that $\chi = \chi(\rho, z)\, \mathrm{d}\phi + \mathrm{d}(\lambda'(\rho, z))$. By a shift in $\psi \to \psi - \lambda'$ (which does not affect $\mathcal{L}_{\partial_\phi}\chi = 0$) we may eliminate λ'. A similar argument works for the 1-forms $\hat{\omega}$ and ξ by shifting t and the gauge field A respectively, establishing the claim. We have thus shown the second part of the lemma.

The last part follows trivially: Putting everything together it is easy to see that the spacetime metric can be be written in the form (2.28) as claimed, and a simple computation shows that indeed $\det G = -\rho^2$, completing the proof of the lemma.

□

It is worth noting that the local form of the metric (2.28) shows the distribution orthogonal to $\mathrm{span}\{\partial_t, \partial_\psi, \partial_\phi\}$ is integrable so that at every point there exist surfaces orthogonal to the Killing fields with metric (2.30). This is equivalent to the Frobenius integrability condition $K_0 \wedge K_1 \wedge K_2 \wedge \mathrm{d}K_A = 0$. [3]

Let us now turn to our global assumptions. As we have already used, we will take solutions to be asymptotically flat. We also assume that for all our solutions, the domain of outer communication $\langle\langle\mathcal{M}\rangle\rangle$ is globally hyperbolic, i.e. $\langle\langle\mathcal{M}\rangle\rangle \cong \mathbb{R} \times \Sigma$. Topological censorship then implies that $\langle\langle\mathcal{M}\rangle\rangle$ is simply connected [46]. We will denote the event horizon by $\mathcal{H}$, although we allow for the possibility of no black hole region. Furthermore, we assume that the supersymmetric Killing field V is complete, so the isometry group is given by $\mathcal{G} = \mathbb{R} \times U(1)^2$, where $\mathbb{R}$ is tangent to the orbits of V. The axes correspond to the set of fixed points of the biaxial symmetry

$$\mathcal{A} = \{p \in \mathcal{M} \mid \det \gamma_{ij}(p) = 0\} \tag{2.33}$$

where γ is the matrix of rotational Killing fields, $\gamma_{ij} = m_i \cdot m_j$ with $i, j = 1, 2$. Under these conditions, as has been discussed in section 1.1.1, it has been shown that the orbit space

$$\hat{\mathcal{M}} = \langle\langle\mathcal{M}\rangle\rangle/\mathcal{G} \cong \Sigma/U(1)^2 \tag{2.34}$$

is a simply connected two-dimensional manifold with boundaries and corners [73, 76, 77]. The axes correspond to boundary segments $I \subset \partial\hat{\mathcal{M}}$ where γ_{ij} is of rank 1 and to corners of $\hat{\mathcal{M}}$ where γ_{ij} is of rank 0. Below we will show that an event horizon, which

[3] This is in fact guaranteed for any solution to the Einstein–Maxwell equation with $D-2$ commuting Killing fields, one of which has at least one fixed point (which must be the case here due to asymptotic flatness), see e.g. [114]. Here it arises as a consequence of supersymmetry which for timelike solutions implies the Einstein–Maxwell equations [53].

must be degenerate, corresponds to a point on $\partial\hat{\mathcal{M}}$ (in fact an asymptotic end, as is generic for extremal horizons, see e.g. [45]).

We may identify surfaces orthogonal to the Killing fields V, m_i with the orbit space $\hat{\mathcal{M}}$. In particular this means that $\hat{\mathcal{M}}$ inherits the metric $q = \frac{H}{f}(\mathrm{d}\rho^2 + \mathrm{d}z^2)$, which we shall refer to as orbit space metric. Under the above global assumptions, it has been shown that $\det G < 0$ everywhere on $\langle\langle\mathcal{M}\rangle\rangle \setminus \mathcal{A}$ and $\det G = 0$ on $\mathcal{H} \cup \mathcal{A}$ [25]. Thus since $\det G = -\rho^2$, we may identify the interior of the orbit space with the upper half plane,

$$\hat{\mathcal{M}} = \{(\rho, z) \mid \rho > 0\} \tag{2.35}$$

and the boundary $\partial\hat{\mathcal{M}}$ and corners with the z-axis ($\rho = 0$). In the orbit space the axes divide into boundary segments $I = (z_i, z_{i+1})$, called axis rods (or intervals), and corners which arise as certain endpoints $z = z_i$ of the axis rods. We will show in the next section that in the orbit space an event horizon is a point on the z-axis. For now let us focus on $\langle\langle\mathcal{M}\rangle\rangle$.

First, on $\langle\langle\mathcal{M}\rangle\rangle \setminus \mathcal{A}$, the orbit space metric must be a smooth, Riemannian manifold, so in particular its conformal factor H/f must be a smooth and positive function,

$$\frac{H}{f} > 0\,. \tag{2.36}$$

Using (2.9) this implies that the invariant $N > 0$. Thus by Lemma 2.1 the harmonic functions H, K, L and M are all smooth on $\langle\langle\mathcal{M}\rangle\rangle \setminus \mathcal{A}$. We also require that the metric is smooth on the axes. Clearly this implies the spacetime invariant $N = f/H$ must be smooth on the axes. We will now show that, away from the corners, N is also positive. Consider the metric near an axis rod $I = (z_i, z_{i+1})$. Then for fixed $z \in I$,

$$\mathrm{d}s^2 = \frac{H}{f}\,\mathrm{d}\rho^2 + G_{tt}\,\mathrm{d}t^2 + 2G_{ti}\,\mathrm{d}t\,\mathrm{d}\phi^i + G_{ij}\,\mathrm{d}\phi^i\,\mathrm{d}\phi^j\,. \tag{2.37}$$

By an $SL(2,\mathbb{Z})$ transformation we may always choose the ϕ_i such that the Killing field vanishing on I is $v = m_1 = \partial_{\phi_1}$. Then for the metric to be smooth at $\rho = 0$, one must have $(\mathrm{d}|m_1|) \to 1$ (otherwise there is a conical singularity at $\rho = 0$). Let us introduce the proper distance from the axis rod, $s = \int_0^\rho \sqrt{g_{\rho\rho}}\,\mathrm{d}\rho$. Then $(\mathrm{d}|m_1|) \to 1$ implies that $G_{11} = s^2(1 + \mathcal{O}(s^2))$, where the order of the subleading term is fixed by the fact that G_{11} must be smooth on the plane with polar coordinates (s, ϕ_1). Furthermore the metric components G_{1A} must vanish smoothly, $G_{1t} = G_{1i} = \mathcal{O}(s^2)$. Therefore

$$-\rho^2 = \det G = \begin{vmatrix} G_{tt} & G_{t2} \\ G_{2t} & G_{22} \end{vmatrix} s^2 + \mathcal{O}(s^4)\,. \tag{2.38}$$

It has been shown that $\mathrm{span}\{K_0, K_1, K_2\}$ is timelike everywhere in $\langle\langle\mathcal{M}\rangle\rangle$ [25]. Therefore, $\mathrm{span}\{V, m_2\}$ must be timelike on I and hence the determinant on the righthand side of (2.38) is strictly negative on I. Hence

$$\frac{f}{H} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 = -\begin{vmatrix} G_{tt} & G_{t2} \\ G_{2t} & G_{22} \end{vmatrix} + \mathcal{O}(s^2) \tag{2.39}$$

is positive on I, so from (2.9) if follows that $N > 0$ and thus by Lemma 2.1 H, K, L and M are all smooth also on an axis rod I.[4]

Thus we arrive at the following crucial result.

Lemma 2.3. *Let $(\mathcal{M}, g, F)$ be an asymptotically flat, supersymmetric and biaxisymmetric solution to minimal supergravity with a globally hyperbolic $\langle\langle\mathcal{M}\rangle\rangle$.*

1. *The fixed points of the triholomorphic Killing field of the Gibbons–Hawking base correspond to precisely the corners of the orbit space $\hat{\mathcal{M}}$.*

2. *The harmonic functions H, K, L, M are smooth and obey (2.16) everywhere in $\langle\langle\mathcal{M}\rangle\rangle$ except possibly at points corresponding to the corners of the orbit space $\hat{\mathcal{M}}$.*

3. *At every corner of the orbit space $f \neq 0$ and H has an isolated singularity.*

Proof. We know that at a corner of the orbit space γ_{ij} is of rank 0, so both axial Killing fields vanish and hence ∂_ψ must as well. Conversely, if ∂_ψ vanishes, from (2.8) it follows that $N = 0$. We have just seen that $N > 0$ in the interior of the orbit space as well as on boundary segments, so $\partial_\psi = 0$ must indeed be a corner. This proves part 1 of the lemma.

To prove part 2 we again use the fact that $N > 0$ everywhere in the orbit space except for the corners (where $N = 0$). Then, by Lemma 2.1, H, K, L and M must be smooth and obey (2.16) everywhere except for possibly at the corners, as claimed.

Finally, to prove the last part of the lemma, recall that $f^2 = -V_\mu V^\mu$. Also, we know that span$\{V, m_1, m_2\}$ must be timelike everywhere in $\langle\langle\mathcal{M}\rangle\rangle$ [25]. Since at a corner the axial Killing fields $m_1 = m_2 = 0$, this implies that V must be timelike at any corner, so in particular $f \neq 0$. Then, as we have seen that $N = 0$ at a corner of $\hat{\mathcal{M}}$, from (2.9) it is clear that H must be singular at the corner. Since the corners are points on the boundary of $\hat{\mathcal{M}}$ and H is smooth everywhere except for the corners, the singularity must be isolated, completing the proof. □

Therefore the analysis reduces to studying the behaviour of the harmonic functions at the event horizon and the corners the of orbit space.

2.4 Near-horizon geometry

We will now turn to a more detailed analysis of the horizon geometry of asymptotically flat, supersymmetric and biaxisymmetric solutions. For any black hole solution, the

[4] Note that (2.38) implies the smoothness condition $\frac{H}{f} = \lim_{\rho\to 0} \frac{\gamma_{ij} v_i v_j}{\rho^2}$ (for $v = v_i m_i$ the Killing vector vanishing on I) previously derived for vacuum Weyl solutions [63, 73, 77].

horizon $\mathcal{H}$ must be a null hypersurface invariant under the isometries of the spacetime, so any Killing vector field must be tangent to the horizon. This implies that the Killing fields must be either null or spacelike at $\mathcal{H}$. In particular, since V is never spacelike, V must be null at $\mathcal{H}$. Hence, $\mathcal{H}$ is a Killing horizon of V. Furthermore, $\mathrm{d}(V\cdot V) = -2f\,\mathrm{d}f$ vanishes on $\mathcal{H}$, so $\mathcal{H}$ has vanishing surface gravity, i.e. it is a degenerate (extreme) horizon. We will generally refer to this as a supersymmetric horizon. Because V is also the stationary Killing field, any such horizon must have vanishing angular velocities, hence is non-rotating.

In the neighbourhood of a supersymmetric horizon we may introduce Gaussian null coordinates (v, λ, x^a) [100],

$$\mathrm{d}s^2 = -\lambda^2\Delta(\lambda,x)^2\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}\lambda + 2\lambda h_a(\lambda,x)\,\mathrm{d}v\,\mathrm{d}x^a + \gamma_{ab}(\lambda,x)\,\mathrm{d}x^a\,\mathrm{d}x^b\,. \tag{2.40}$$

Here $\partial_v = V$ is the Killing vector field normal to the horizon, x^a are coordinates on a cross-section of the horizon (with metric γ_{ab}) and, ∂_λ is defined as the vector field tangent to the unique past directed null geodesic such that $\partial_v\cdot\partial_\lambda = 1$ and $\partial_\lambda\cdot\partial_a = 0$. In particular, while a priori the coordinates x^a are only defined on the horizon, they may be extended into a neighbourhood of $\mathcal{H}$ via Lie transport along the integral curves of ∂_λ. The horizon is located at $\lambda = 0$ and $f = \lambda\Delta(\lambda, x)$. The supersymmetric near-horizon geometries of minimal supergravity have been classified [100]: Assuming cross-sections of the horizon are compact, it can be shown that $\Delta|_{\lambda=0} = \Delta_0$ is a constant on the horizon. If $\Delta_0 \neq 0$ the horizon is locally S^3, and if $\Delta_0 = 0$ it is $S^1\times S^2$ (we do not consider the T^3 case as this is not an allowed topology for black holes)[5]. In fact, an output of this analysis is that the near-horizon geometry must have biaxial symmetry. We will use this classification below after a general analysis of the orbit space.

2.4.1 Orbit space metric

Let us now consider the orbit space metric near an extreme horizon. Our analysis in this section will be general and only assume the existence of biaxial symmetry.

Since the biaxial Killing fields must be tangent to the horizon, we may always choose Gaussian null coordinates adapted to this symmetry, $(v, \lambda, \tilde\theta, \tilde\phi^i)$ where $(\tilde\theta, \tilde\phi^i)$, $i = 1, 2$, are such that the three commuting Killing fields are $V = \partial_v$, $m_i = \partial_{\tilde\phi^i}$. Then we write

$$\begin{aligned}\mathrm{d}s^2 = -\lambda^2 F\,\mathrm{d}v^2 + 2\,\mathrm{d}v(\mathrm{d}\lambda + \lambda\tilde h_{\tilde\theta}\,\mathrm{d}\tilde\theta) + \gamma_{\tilde\theta\tilde\theta}\,\mathrm{d}\tilde\theta^2 + 2\gamma_{\tilde\theta i}\,\mathrm{d}\tilde\theta(\mathrm{d}\tilde\phi^i + \lambda h^i\,\mathrm{d}v)\\ + \gamma_{ij}(\mathrm{d}\tilde\phi^i + \lambda h^i\,\mathrm{d}v)(\mathrm{d}\tilde\phi^j + \lambda h^j\,\mathrm{d}v)\,,\end{aligned} \tag{2.41}$$

where $F = \Delta^2 + h^i h_i$, $\tilde h_{\tilde\theta} = h_{\tilde\theta} - \gamma_{\tilde\theta i}h^i$ and $h^i = \gamma^{ij}h_j$ where γ^{ij} is the inverse of the 2d

[5] The analysis of [100] assumes $f > 0$ (i.e. $\Delta > 0$) for small $r > 0$. However, this is restrictive and we should only assume $f \neq 0$ (i.e. $\Delta \neq 0$ for small $r > 0$). In fact, the analysis of [100] remains valid under this weaker assumption since only Δ^2 appears in the near-horizon geometry.

matrix γ_{ij}. Then the inner product on the space of Killing fields is

$$G = -\lambda^2 F\,\mathrm{d}v^2 + \gamma_{ij}(\mathrm{d}\tilde{\phi}^i + \lambda h^i\,\mathrm{d}v)(\mathrm{d}\tilde{\phi}^j + \lambda h^j\,\mathrm{d}v) \tag{2.42}$$

with inverse

$$G^{AB} = \begin{pmatrix} -\frac{1}{\lambda^2 F} & \frac{h^i}{\lambda F} \\ \frac{h^j}{\lambda F} & \gamma^{ij} - \frac{h^i h^j}{F} \end{pmatrix} . \tag{2.43}$$

As the spacetime has three commuting Killing fields, the distribution orthogonal to these Killing fields is integrable and there exist local coordinates in which the metric takes block diagonal form (2.28) (see e.g. [114]). To bring the metric into this form, we make the coordinate transformations

$$t = v + A(\lambda, \tilde{\theta})\,, \qquad \phi^i = \tilde{\phi}^i + B^i(\lambda, \tilde{\theta})\,, \tag{2.44}$$

where $V = \partial_t$, $m_i = \partial_{\phi^i}$ and

$$\partial_\lambda A = -\frac{1}{\lambda^2 F}, \qquad \partial_{\tilde{\theta}} A = -\frac{\tilde{h}_{\tilde{\theta}}}{\lambda F}, \qquad \partial_\lambda B^i = \frac{h^i}{\lambda F}, \qquad \partial_{\tilde{\theta}} B^i = \gamma^{ij}\gamma_{\tilde{\theta}j} + \frac{h^i \tilde{h}_{\tilde{\theta}}}{F}\,. \tag{2.45}$$

From this one finds the orbit space metric

$$q = \frac{1}{F\lambda^2}(\mathrm{d}\lambda + \lambda \tilde{h}_{\tilde{\theta}}\,\mathrm{d}\tilde{\theta})^2 + (\gamma_{\tilde{\theta}\tilde{\theta}} - \gamma^{ij}\gamma_{i\tilde{\theta}}\gamma_{j\tilde{\theta}})\,\mathrm{d}\tilde{\theta}^2\,, \tag{2.46}$$

with the matrix of Killing fields in the new coordinates given by

$$G = -\lambda^2 F\,\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}\phi^i + \lambda h^i\,\mathrm{d}t)(\mathrm{d}\phi^j + \lambda h^j\,\mathrm{d}t)\,. \tag{2.47}$$

Alternatively, one could derive the orbit space metric following [27] from the matrix of Killing fields in the original coordinates, (2.42), (2.43) and defining q via

$$q_{\mu\nu} = g_{\mu\nu} - G^{AB} g_{A\mu} g_{B\nu}\,, \tag{2.48}$$

leading again to (2.46).

We will now examine the orbit space metric near the horizon. We will assume $h_{\tilde{\theta}} = \mathcal{O}(\lambda)$ and $\gamma_{i\tilde{\theta}} = \mathcal{O}(\lambda)$ which are conditions satisfied by the near-horizon geometries in question, see [100]. We find that near the horizon (2.46) takes the form

$$q = \left(\frac{1}{F\lambda^2} + \mathcal{O}(\lambda^{-1})\right)\mathrm{d}\lambda^2 + \mathcal{O}(1)\,\mathrm{d}\lambda\,\mathrm{d}\tilde{\theta} + (\gamma_{\tilde{\theta}\tilde{\theta}} + \mathcal{O}(\lambda))\,\mathrm{d}\tilde{\theta}\,, \tag{2.49}$$

and the determinant of the Killing metric is

$$\begin{aligned} \rho &= \sqrt{-\det G} = \lambda\sqrt{F\det\gamma_{ij}} \\ &= \sqrt{F\det\gamma_{ij}}|_{\lambda=0}\,\lambda + \mathcal{O}(\lambda^2)\,. \end{aligned} \tag{2.50}$$

The function ρ is harmonic in the orbit space metric (2.30). The harmonic conjugate z is given by $\mathrm{d}z = \star_2 \mathrm{d}\rho$. A computation gives

$$\begin{aligned}
\partial_\lambda z &= \frac{1}{\lambda\sqrt{F q_{\tilde\theta\tilde\theta}}}\left(\partial_{\tilde\theta}\rho - \lambda \tilde h_{\tilde\theta}\partial_\lambda \rho\right) = \left.\frac{\partial_{\tilde\theta}\sqrt{F\det\gamma_{ij}}}{\sqrt{F\gamma_{\tilde\theta\tilde\theta}}}\right|_{\lambda=0} + \mathcal{O}(\lambda)\,,\\
\partial_{\tilde\theta} z &= \frac{1}{\sqrt{F q_{\tilde\theta\tilde\theta}}}\left[-\lambda F q_{\tilde\theta\tilde\theta}\partial_\lambda\rho + \tilde h_{\tilde\theta}\left(\partial_{\tilde\theta}\rho - \lambda\tilde h_{\tilde\theta}\partial_\lambda\rho\right)\right] = -\left.F\sqrt{\gamma_{\tilde\theta\tilde\theta}\det\gamma_{ij}}\right|_{\lambda=0}\lambda + \mathcal{O}(\lambda^2)\,.
\end{aligned} \tag{2.51}$$

Observe that by integrating for z we see that it will take the form $z = z_0 + \mathcal{O}(\lambda)$, where z_0 is a constant which can be set to zero. Hence, a degenerate horizon corresponds to a single point on the boundary of the orbit space. In fact, in the orbit space metric, it is easy to see from $q_{\lambda\lambda}$ that any point is an infinite proper distance to the horizon, so that a degenerate horizon corresponds to an asymptotic end (see [45] for discussion of this in the vacuum case).

To proceed further we need the specific near-horizon geometries. We will turn to this next.

2.4.2 Locally S^3 horizon

For a locally S^3 horizon, the horizon data is given by [100]

$$\begin{aligned}
\Delta_0^2 = \frac{4}{\mu}\left(1-\frac{j^2}{\mu^3}\right)\,, \qquad h|_{\lambda=0} &= -j\mu^{-3/2}\left(1-\frac{j^2}{\mu^3}\right)^{1/2}(\mathrm{d}\tilde\psi + \cos\tilde\theta\,\mathrm{d}\tilde\phi)\,,\\
\gamma|_{\lambda=0} = \frac{\mu}{4}\left[\left(1-\frac{j^2}{\mu^3}\right)(\mathrm{d}\tilde\psi + \cos\tilde\theta\,\mathrm{d}\tilde\phi)^2 + \mathrm{d}\tilde\theta^2 + \sin^2\tilde\theta\,\mathrm{d}\tilde\phi^2\right]\,,&
\end{aligned} \tag{2.52}$$

where the constants $j^2 < \mu^3$. We deduce that near the horizon the metric is given by (2.41) where

$$\begin{aligned}
F = \frac{4}{\mu} + \mathcal{O}(\lambda), \qquad h^i &= -\frac{4j}{\mu^{5/2}\left(1-\frac{j^2}{\mu^3}\right)^{1/2}}\delta^i_{\tilde\psi} + \mathcal{O}(\lambda)\,,\\
h_{\tilde\theta} = \mathcal{O}(\lambda), \qquad \gamma_{\tilde\theta\tilde\theta} &= \frac{\mu}{4} + \mathcal{O}(\lambda), \qquad \gamma_{\tilde\theta i} = \mathcal{O}(\lambda)\,,\\
\gamma_{ij}\,\mathrm{d}\tilde\phi^i\,\mathrm{d}\tilde\phi^j = \frac{\mu}{4}&\left[\left(1-\frac{j^2}{\mu^3}\right)(\mathrm{d}\tilde\psi + \cos\tilde\theta\,\mathrm{d}\tilde\phi)^2 + \sin^2\tilde\theta\,\mathrm{d}\tilde\phi^2\right] + \mathcal{O}(\lambda)\,.
\end{aligned} \tag{2.53}$$

The above horizon geometry is locally that of a squashed S^3 with a $U(1)^2$-isometry generated by the Killing fields $(\partial_{\tilde\psi}, \partial_{\tilde\phi})$. The topology of the horizon is determined by the periodicity lattice of the biaxial Killing fields. For now our analysis will be local, but we will consider global constraints at the end of this section. In any case, locally, biaxial symmetry implies that $(\partial_{\tilde\psi}, \partial_{\tilde\phi})$ must be related to the Gibbons–Hawking space biaxial Killing fields $(\partial_\psi, \partial_\phi)$ by a constant linear transformation. Hence, $\det\gamma_{ij}|_{i,j=\tilde\psi,\tilde\phi} = c^2\det\gamma_{ij}|_{i,j=\psi,\phi}$ for some constant $c > 0$. We need to take account of this Jacobian when comparing the determinants of the matrix of Killing fields to determine the Weyl coordinates (ρ, z).

Using the near horizon data (2.53), we find that (2.50) and (2.51) imply

$$\rho = \sqrt{\frac{\mu}{4c^2}\left(1-\frac{j^2}{\mu^3}\right)}\lambda \sin\tilde{\theta} + \mathcal{O}(\lambda^2)\,, \qquad z = \sqrt{\frac{\mu}{4c^2}\left(1-\frac{j^2}{\mu^3}\right)}\lambda \cos\tilde{\theta} + \mathcal{O}(\lambda^2)\,. \tag{2.54}$$

Thus the $\mathbb{R}^3$ polar coordinates are

$$r = \sqrt{\frac{\mu}{4c^2}\left(1-\frac{j^2}{\mu^3}\right)}\lambda + \mathcal{O}(\lambda^2)\,, \qquad \cos\theta = \cos\tilde{\theta} + \mathcal{O}(\lambda)\,. \tag{2.55}$$

We deduce that the horizon is a single point in the orbit space metric, as anticipated above. The orbit space metric in the (ρ, z) coordinates must take the form (2.30), so using the coordinate change (2.54) we find

$$q = \frac{H}{f}\left[\frac{\mu}{4c^2}\left(1-\frac{j^2}{\mu^3}\right)(\mathrm{d}\lambda^2 + \lambda^2\,\mathrm{d}\tilde{\theta}^2) + \mathcal{O}(\lambda)\,\mathrm{d}\lambda^2 + \mathcal{O}(\lambda^2)\,\mathrm{d}\lambda\,\mathrm{d}\tilde{\theta} + \mathcal{O}(\lambda^3)\,\mathrm{d}\tilde{\theta}^2\right] \tag{2.56}$$

and comparing to (2.49) implies

$$\frac{H}{f} = \frac{c^2}{1-\frac{j^2}{\mu^3}}\left(\frac{1}{\lambda^2} + \mathcal{O}(\lambda^{-1})\right)\,. \tag{2.57}$$

Therefore, using $f = \lambda\Delta_0 + \mathcal{O}(\lambda^2)$ we deduce

$$H = \frac{4c^2}{\mu\Delta_0\lambda} + \mathcal{O}(1) = \frac{h_0}{r} + \mathcal{O}(1)\,, \tag{2.58}$$

where we have defined the constant $h_0 = \mathrm{sgn}(\Delta_0)c$. Observe that the first term in H is harmonic and hence the $\mathcal{O}(1)$ term is also a harmonic function. Therefore, we deduce that a horizon corresponds to an isolated singularity of the harmonic function H; specifically the horizon is a pole of order one.

We now turn to the remaining harmonic functions. Firstly, observe that we can write $\partial_\psi = a\partial_{\tilde{\psi}} + b\partial_{\tilde{\phi}}$ for some constants a, b. Hence the invariant $g_{t\psi} = V\cdot\partial_\psi = \lambda h\cdot\partial_\psi = \lambda(ah_{\tilde{\psi}} + bh_{\tilde{\phi}})$. Then, the near-horizon expansion of the invariants $g_{t\psi}$ and f, together with (2.11) and smoothness of Ψ, imply that KH^{-1} is smooth at $\lambda = 0$. Therefore $K = \mathcal{O}(r^{-1})$ and since it is harmonic it must have a pole of order one, so

$$K = \frac{k_0}{r} + \mathcal{O}(1)\,, \tag{2.59}$$

where k_0 is a constant and the $\mathcal{O}(1)$ term is harmonic. Due to the shift freedom in K (2.14) the constant k_0 can be set to any value.

Next, using the expansion of the invariants f and $-f^2\omega_\psi = V\cdot\partial_\psi$ together with

(1.94) and (1.95) implies

$$L=\frac{\ell_0}{r}+\mathcal{O}(1), \qquad M=\frac{1}{r}\left(\frac{j(a+b\cos\theta)}{8c}+\frac{k_0^3}{2h_0^2}-\frac{3\mu k_0}{8h_0^2}\right)+\mathcal{O}(1)\,, \tag{2.60}$$

where $\ell_0=h_0^{-1}(\frac{1}{4}\mu - k_0^2)$ and hence L has a pole of order one. Also we have $M=\mathcal{O}(r^{-1})$ and since it is harmonic this implies M also has a pole of order one so must be of the form $M=m_0/r+\mathcal{O}(1)$ where m_0 is a constant. Hence $b=0$ so we deduce the triholomorphic Killing field $\partial_\psi = a\partial_{\tilde{\psi}}$. In summary, so far we have shown that a regular horizon corresponds to a simple pole of the harmonic functions H, K, L, M.

We now turn to global constraints. The precise periodicities of $(\tilde{\psi},\tilde{\phi})$ determine the horizon topology which in general may be that of a lens space. Now, asymptotic flatness fixes the identifications of the Gibbons–Hawking space angles (ψ,ϕ) to be standard Euler angles on S^3 (2.24). This will impose identifications on the $(\tilde{\psi},\tilde{\phi})$ angles. To analyse this, it is convenient to note that the Killing vectors on the horizon which have fixed points at the poles $\tilde{\theta}=0,\pi$ are $\tilde{v}_\pm = \partial_{\tilde{\phi}} \mp \partial_{\tilde{\psi}}$. For a lens space $L(p,q)$ these must be related to the independently 2π-periodic vectors (2.24) of the asymptotically flat region by

$$\begin{pmatrix} \tilde{v}_- \\ \tilde{v}_+ \end{pmatrix} = A \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \tag{2.61}$$

where $A\in GL(2,\mathbb{Z})$ and $\det A = p \in \mathbb{Z}$. The corresponding transformation in terms of Euler angles can be deduced from (2.24), which implies $\det \gamma_{ij}|_{i,j=\tilde{\psi},\tilde{\phi}} = p^2 \det \gamma_{ij}|_{i,j=\psi,\phi}$. Thus, comparing to our local analysis above shows that $c=\pm p$ is precisely the integer which defines the lens spaces $L(p,q)$. In fact, by fixing the sign of p appropriately we will identify the constant in (2.58) as

$$h_0 = p\,. \tag{2.62}$$

Equations (2.58)–(2.60) derived in this section are necessary conditions for regularity at the horizon. We will examine sufficient conditions for a regular horizon in section 2.4.4.

2.4.3 $S^1\times S^2$ horizon

We will now repeat the above analysis for the other type of near-horizon geometry. The horizon data is now

$$\begin{aligned} \Delta_0 = 0\,, \qquad h|_{\lambda=0} &= \frac{R}{\ell}\,\mathrm{d}\tilde{\psi}\,, \\ \gamma|_{\lambda=0} &= R^2\,\mathrm{d}\tilde{\psi}^2 + \ell^2(\mathrm{d}\tilde{\theta}^2+\sin^2\tilde{\theta}\,\mathrm{d}\tilde{\phi}^2)\,, \end{aligned} \tag{2.63}$$

where the constant $R > 0$ has been introduced for later convenience. Thus, near the horizon the metric is given by (2.41) where

$$\begin{gathered} F = \frac{1}{\ell^2} + \mathcal{O}(\lambda)\,, \qquad h^i = \frac{1}{R\ell}\delta^i_{\tilde{\psi}} + \mathcal{O}(\lambda)\,, \\ h_{\tilde{\theta}} = \mathcal{O}(\lambda)\,, \qquad \gamma_{\tilde{\theta}\tilde{\theta}} = \ell^2 + \mathcal{O}(\lambda)\,, \qquad \gamma_{\tilde{\theta}i} = \mathcal{O}(\lambda)\,, \\ \gamma_{ij}\,\mathrm{d}\tilde{\phi}^i\,\mathrm{d}\tilde{\phi}^j = R^2\,\mathrm{d}\tilde{\psi}^2 + \ell^2\sin^2\tilde{\theta}\,\mathrm{d}\tilde{\phi}^2 + \mathcal{O}(\lambda)\,. \end{gathered} \tag{2.64}$$

The near-horizon geometry has biaxial symmetry generated by the Killing fields $\partial_{\tilde{\psi}}, \partial_{\tilde{\phi}}$. As before, these must be related by a constant linear transformation to the biaxial Killing fields of the Gibbons–Hawking space, so $\det \gamma_{ij}|_{i,j=\tilde{\psi},\tilde{\phi}} = c^2 \det \gamma_{ij}|_{i,j=\psi,\phi}$ for some constant $c > 0$. Then, using (2.50) and (2.51) we find

$$\rho = \frac{R}{c}\lambda\sin\tilde{\theta} + \mathcal{O}(\lambda^2)\,, \qquad z = \frac{R}{c}\lambda\cos\tilde{\theta} + \mathcal{O}(\lambda^2)\,, \tag{2.65}$$

so the $\mathbb{R}^3$ polar coordinates are

$$r = \frac{R}{c}\lambda + \mathcal{O}(\lambda^2)\,, \qquad \cos\theta = \cos\tilde{\theta} + \mathcal{O}(\lambda)\,. \tag{2.66}$$

Hence, again, the horizon corresponds to a point in the orbit space metric.

We may now compare to the orbit space metric (2.30) and (2.49) near the horizon. Using (2.65) we find

$$\frac{H}{f} = \frac{c^2\ell^2}{R^2\lambda^2} + \mathcal{O}(\lambda^{-1}) = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-1})\,. \tag{2.67}$$

The function $f = \lambda^2\tilde{\Delta} + \mathcal{O}(\lambda^3)$, for some non-zero constant $\tilde{\Delta}$ [100]. Thus we learn that

$$H = \frac{\tilde{\Delta}\ell^2}{R^2} + \mathcal{O}(\lambda)\,, \tag{2.68}$$

so in this case H is a smooth harmonic function at the horizon.

We now determine the other harmonic functions. Near the horizon the invariant $g_{t\psi} = V\cdot\partial_\psi = \lambda h\cdot\partial_\psi = \lambda a h_{\tilde{\psi}} + \mathcal{O}(\lambda^2)$, where the final equality follows from writing $\partial_\psi = a\partial_{\tilde{\psi}} + b\partial_{\tilde{\phi}}$ for constants a, b. Then, expanding (2.11) near the horizon we find

$$KH^{-1} = -\frac{aR}{\ell\tilde{\Delta}\lambda} + \mathcal{O}(1)\,, \tag{2.69}$$

where we have used the near-horizon expansion of f and smoothness of Ψ. Hence

$$K = -\frac{a\ell}{R\lambda} + \mathcal{O}(1) = -\frac{a\ell}{cr} + \mathcal{O}(1)\,. \tag{2.70}$$

Using the invariants f and $V\cdot\partial_\psi = -f^2\omega_\psi$, together with (1.94) and (1.95), then implies

$$L = \frac{(1-a^2)R^2}{\tilde{\Delta} c^2 r^2} + \mathcal{O}(r^{-1}), \qquad M = \frac{a(1-a^2)R^4}{\tilde{\Delta}\ell c^3 r^3} + \mathcal{O}(r^{-2})\,. \tag{2.71}$$

Therefore L has a pole of order at most two. However, a harmonic function in $\mathbb{R}^3$ with a pole of order two must be of the form $L = c_1 r^{-2}\cos\theta + c_2 r^{-1} + \mathcal{O}(1)$. Thus, since the coefficient of the r^{-2} term is a constant it must vanish and hence we have $a = 1$ (choosing a sign). Hence $L = \mathcal{O}(r^{-1})$, so harmonicity implies it has a pole of at most order one. This then also implies M has a pole of at most order two.

In fact we may show that $M = \mathcal{O}(r^{-1})$ as follows. The explicit expression for the invariant $g_{\psi\psi}$ in (2.1), together the above behaviour of the harmonic functions requires $K^3M/(HL+K^2)^2$ to be smooth at $\lambda = 0$. It follows that

$$M = \mathcal{O}(r^{-1}) \tag{2.72}$$

and hence harmonicity implies it must also have a pole of order one.

We now turn to global constraints imposed by asymptotic flatness. In the case of a $S^1 \times S^2$ horizon, $\partial_{\tilde{\psi}}$ and $\partial_{\tilde{\phi}}$ are independently periodic and we will choose the constant R so that $\tilde{\psi}$ has period 4π. Hence we can relate $2\partial_{\tilde{\psi}}$ and $\partial_{\tilde{\phi}}$ to the independently 2π periodic vectors v_+, v_- of the asymptotically flat region (2.24) by a $SL(2,\mathbb{Z})$ transformation. Their relation to the Euler angles of the Gibbons–Hawking base can then be deduced from (2.24). We find $\det \gamma_{ij}|_{i,j=\tilde{\psi},\tilde{\phi}} = \det \gamma_{ij}|_{i,j=\psi,\phi}$ so we deduce the constant $c = 1$.

We will examine sufficient conditions for regularity of the horizon in section 2.4.4.

2.4.4 Horizon regularity and topology

As we have seen above a regular horizon corresponds to an isolated singularity in the $\mathbb{R}^3$ base of the Gibbons–Hawking space which we may take to be the origin $r = 0$. Furthermore, the harmonic functions have at most simple poles at the horizon and so can be written as

$$H = \frac{h_0}{r} + H_0\,, \qquad K = \frac{k_0}{r} + K_0\,, \qquad L = \frac{\ell_0}{r} + L_0\,, \qquad M = \frac{m_0}{r} + M_0\,, \tag{2.73}$$

where h_0, k_0, ℓ_0, m_0 are constants and H_0, K_0, L_0, M_0 are harmonic functions that are smooth at $r = 0$. Thus we can write $H_0 = c_0 + \mathcal{O}(r)$, where c_0 is a constant and the $\mathcal{O}(r)$ term is analytic in r, and similarly for the other harmonic functions. In particular, using (2.1) this implies

$$\begin{aligned} \frac{H}{f} &= \frac{\alpha_0}{r^2} + \frac{\alpha_1}{r} + \mathcal{O}(1)\,, \\ g_{\psi\psi} &= \beta_0 + r\beta_1 + \mathcal{O}(r^2)\,, \\ g_{t\psi} &= r(\gamma_0 + r\gamma_1 + \mathcal{O}(r^2))\,, \end{aligned} \tag{2.74}$$

where $\alpha_i, \beta_i, \gamma_i$ are *constants* and comparing to the near-horizon analysis in the previous sections implies

$$\alpha_0 > 0, \qquad \beta_0 > 0\,. \tag{2.75}$$

Then we can write

$$f = r\left(\frac{h_0}{\alpha_0} + \frac{c_0\alpha_0 - h_0\alpha_1}{\alpha_0^2} r + \mathcal{O}(r^2)\right)\,. \tag{2.76}$$

The explicit expressions for leading order coefficients are

$$\begin{aligned} \alpha_0 &= h_0\ell_0 + k_0^2\,, \\ \beta_0 &= \frac{-h_0^2 m_0^2 - 3h_0 k_0 \ell_0 m_0 + h_0 \ell_0^3 - 2k_0^3 m_0 + \frac{3}{4}k_0^2\ell_0^2}{\alpha_0^2}\,, \\ \gamma_0^2 &= \frac{\alpha_0 - h_0^2\beta_0}{\alpha_0^2} = \left(\frac{h_0^2 m_0 + \frac{3}{2}h_0 k_0 \ell_0 + k_0^3}{\alpha_0^2}\right)^2\,. \end{aligned} \tag{2.77}$$

Notice that the last relation shows that $\gamma_0^2 \geq 0$ does not lead to any inequalities beyond (2.75). In fact, from the very same relation we can see that $\alpha_0^2\beta_0 > 0$ actually implies $\alpha_0 > 0$ so (2.75) is really equivalent to the single condition

$$-h_0^2 m_0^2 - 3h_0 k_0 \ell_0 m_0 + h_0\ell_0^3 - 2k_0^3 m_0 + \frac{3}{4}k_0^2\ell_0^2 > 0 \tag{2.78}$$

on the parameters h_0, k_0, ℓ_0, m_0. It is worth noting that

$$K^2 + HL = \frac{\alpha_0}{r^2} + \mathcal{O}(r^{-1}), \qquad g^{tt} = -\frac{\alpha_0\beta_0}{r^2} + \mathcal{O}(r^{-1}) \tag{2.79}$$

which already confirms that the above inequalities imply the solution is smooth (2.16) and stably causal (2.18) near (but not at) the horizon. We will now show that (2.78) is sufficient for regularity at the horizon.

To this end, let us perform a coordinate transformation

$$\mathrm{d}t = \mathrm{d}v + \left(\frac{A_0}{r^2} + \frac{A_1}{r}\right)\mathrm{d}r\,, \qquad \mathrm{d}\psi = \mathrm{d}\psi' + \frac{B_0}{r}\,\mathrm{d}r + C\,\mathrm{d}\phi'\,, \qquad \mathrm{d}\phi = \mathrm{d}\phi'\,, \tag{2.80}$$

where A_0, A_1, B_0, C are constants to be determined. Using the above expansions, it follows that g_{rr} contains $1/r^2$ and $1/r$ singular terms, whereas $g_{r\psi'}$ contains $1/r$ singular terms. Demanding that the $1/r^2$ term in g_{rr} and $1/r$ term in $g_{t\psi'}$ vanish is equivalent to setting

$$A_0^2 = \beta_0\alpha_0^2, \qquad B_0 = -\frac{A_0\gamma_0}{\beta_0}\,. \tag{2.81}$$

Demanding that the $1/r$ term in g_{rr} vanishes, fixes

$$A_1 = \frac{\alpha_0\beta_0}{2A_0}\left(B_0^2\beta_1 + 2B_0A_0\gamma_1 + \alpha_1 - \frac{2h_0(c_0\alpha_0 - h_0\alpha_1)}{\alpha_0^3}A_0^2\right)\,. \tag{2.82}$$

Note that we have simplified A_0, A_1 using the identity for γ_0 above. With these choices,

g_{rr} and $g_{r\psi'}$ are analytic at $r = 0$.

We will also need the near-horizon behaviour of the 1-forms χ, $\hat{\omega}$, ξ. Using the behaviour of the harmonic functions (2.73) near the horizon we find

$$\star_3 \mathrm{d}\chi = \big(- h_0 r^{-2} + \mathcal{O}(1) \big) \, \mathrm{d}r + \mathcal{O}(r) \, \mathrm{d}\theta \,, \tag{2.83}$$

$$\star_3 \mathrm{d}\hat{\omega} = \mathcal{O}(r^{-2}) \, \mathrm{d}r + \mathcal{O}(1) \, \mathrm{d}\theta \,, \tag{2.84}$$

$$\star_3 \mathrm{d}\xi = \big(k_0 r^{-2} + \mathcal{O}(1) \big) \, \mathrm{d}r + \mathcal{O}(r) \, \mathrm{d}\theta \,, \tag{2.85}$$

and writing the 1-forms as (2.27), we may integrate to get

$$\chi = \big(h_0 \cos\theta + \tilde{\chi}_0 + \mathcal{O}(r^2) \big) \, \mathrm{d}\phi \,, \qquad \hat{\omega} = \mathcal{O}(1) \, \mathrm{d}\phi \,, \qquad \xi = \big(- k_0 \cos\theta + \tilde{\xi}_0 + \mathcal{O}(r^2) \big) \, \mathrm{d}\phi \tag{2.86}$$

for some constants $\tilde{\chi}_0$, $\tilde{\xi}_0$. For convenience we choose $C = -\tilde{\chi}_0$ in (2.80). The full metric near $r = 0$ now reads

$$\begin{aligned} \mathrm{d}s^2 = & -r^2 \Big(\frac{h_0^2}{\alpha_0^2} + \mathcal{O}(r) \Big) \big(\mathrm{d}v + \mathcal{O}(1) \, \mathrm{d}\phi \big)^2 \pm 2 \Big(\frac{1}{\sqrt{\beta_0}} + \mathcal{O}(r) \Big) \big(\mathrm{d}v + \mathcal{O}(1) \, \mathrm{d}\phi \big) \, \mathrm{d}r + \mathcal{O}(1) \, \mathrm{d}r^2 \\ & + \mathcal{O}(1) \big(\mathrm{d}\psi' + h_0 \cos\theta \, \mathrm{d}\phi' + \mathcal{O}(r^2) \big) \, \mathrm{d}r + 2r \big(\gamma_0 + \mathcal{O}(r) \big) \big(\mathrm{d}v + \mathcal{O}(1) \, \mathrm{d}\phi \big) \big(\mathrm{d}\psi' + h_0 \cos\theta \, \mathrm{d}\phi' \big) \\ & + \big(\beta_0 + \mathcal{O}(r) \big) \big(\mathrm{d}\psi' + h_0 \cos\theta \, \mathrm{d}\phi' + \mathcal{O}(r^2) \big)^2 + \big(\alpha_0 + \mathcal{O}(r) \big) \big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi'^2 \big) \,. \end{aligned} \tag{2.87}$$

The metric and its inverse are now analytic at $r = 0$, hence the spacetime can be analytically extended to the region $r \leq 0$. The surface $r = 0$ is an extremal Killing horizon with respect to the supersymmetric Killing field $V = \partial/\partial v$. The upper (lower) sign in g_{vr} corresponds to a future (past) horizon. The gauge field near $r = 0$ is given by

$$\begin{aligned} A = \frac{\sqrt{3}}{2} \bigg[& \Big(\frac{h_0}{\alpha_0} r + \mathcal{O}(r^2) \Big) \, \mathrm{d}v \pm \Big(\frac{\beta_0 h_0 - \gamma_0 (h_0 m_0 + \frac{1}{2} k_0 \ell_0)}{\sqrt{\beta_0} r} + \mathcal{O}(1) \Big) \, \mathrm{d}r \\ & + \Big(\frac{h_0 m_0 + \frac{1}{2} k_0 \ell_0}{\alpha_0} + \mathcal{O}(r) \Big) (\mathrm{d}\psi' + h_0 \cos\theta \, \mathrm{d}\phi') - \Big(\tilde{\xi}_0 - k_0 \cos\theta + \mathcal{O}(r) \Big) \, \mathrm{d}\phi' \bigg] \,, \end{aligned} \tag{2.88}$$

so we see that the only singular terms are pure gauge. Therefore the Maxwell field $F = \mathrm{d}A$ (and hence the full solution) is analytic at $r = 0$.

The near-horizon limit can be taken by transforming to coordinates $(v, r) \to (v/\epsilon, \epsilon r)$ and letting $\epsilon \to 0$, giving the near-horizon geometry

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{NH}} = & -r^2 \frac{h_0^2}{\alpha_0^2} \, \mathrm{d}v^2 \pm \frac{2}{\sqrt{\beta_0}} \, \mathrm{d}v \, \mathrm{d}r + 2r\gamma_0 \, \mathrm{d}v (\mathrm{d}\psi' + h_0 \cos\theta \, \mathrm{d}\phi') \\ & + \beta_0 (\mathrm{d}\psi' + h_0 \cos\theta \, \mathrm{d}\phi')^2 + \alpha_0 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi'^2) \,, \end{aligned} \tag{2.89}$$

as well as the near-horizon Maxwell field

$$F_{\mathrm{NH}} = \frac{\sqrt{3}}{2}\left[\frac{h_0}{\alpha_0}\,\mathrm{d}r\wedge\mathrm{d}v - \left(\frac{h_0}{\alpha_0}(h_0 m_0 + \tfrac{1}{2}k_0\ell_0) + k_0\right)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi'\right]. \tag{2.90}$$

The second line in (2.89) is the metric induced on cross-sections of the horizon. For $h_0 = 0$ it is simply the standard product metric on $S^1\times S^2$. For $h_0\neq 0$ it is a locally homogeneous metric on S^3.

Our analysis so far in this section has been local. We will now examine the constraints imposed by asymptotic flatness. Recall in our near-horizon analysis in section 2.4.2 and 2.4.3 we showed that $h_0 = p\in\mathbb{Z}$ is the integer which fixes the horizon topology to be a lens space $L(p,q)$. The precise topology is determined by the identifications on the angles. These are already fixed by asymptotic flatness which requires (ψ,ϕ) to be identified as the standard Euler angles on S^3. For $p\neq 0$, it is convenient to define $\bar\psi = \psi'/p$ and $\bar\phi = \phi'$. From the coordinate change (2.80), the Killing fields are related by

$$\begin{pmatrix}\partial_{\bar\psi}\\ \partial_{\bar\phi}\end{pmatrix} = \begin{pmatrix}p & 0\\ -\tilde\chi_0 & 1\end{pmatrix}\begin{pmatrix}\partial_\psi\\ \partial_\phi\end{pmatrix} \tag{2.91}$$

and hence the matrix A in (2.61) is determined using (2.24). Requiring the entries of A to be integer is then equivalent to $\tilde\chi_0 = p+2n-1$ for some integer n. The matrix A then simplifies to

$$A = \begin{pmatrix}1-n & n\\ -p-n+1 & p+n\end{pmatrix}. \tag{2.92}$$

By a basis change $A\to A' = AB$ where $B\in SL(2,\mathbb{Z})$ we can bring the matrix into triangular form

$$A' = \begin{pmatrix}1 & 0\\ q & p\end{pmatrix}, \tag{2.93}$$

where

$$B = \begin{pmatrix}\alpha & -n\\ \beta & 1-n\end{pmatrix} \tag{2.94}$$

and $(1-n)\alpha + n\beta = 1$ and $q = 1 + p(\beta-\alpha)$. We deduce the important result

$$q\equiv 1 \mod p\,. \tag{2.95}$$

Therefore, we have shown that the identifications that arise from asymptotic flatness, together with regularity, imply the only possible horizon topology is $L(p,1)$. With these global identifications we find the area of cross-sections of the horizon is

$$A = 16\pi^2\sqrt{-h_0^2m_0^2 - 3h_0k_0\ell_0m_0 + h_0\ell_0^3 - 2k_0^3m_0 + \frac{3}{4}k_0^2\ell_0^2}\,, \tag{2.96}$$

where the expression under the square root is positive (2.78). This completes our analysis of the horizon.

2.4.5 Summary

To summarise, we have established the following results.

Theorem 2.2. *Consider a supersymmetric and biaxisymmetric solution to minimal supergravity containing a smooth supersymmetric horizon with compact cross-sections of topology $S^1 \times S^2$ or locally S^3. In the orbit space metric the horizon is an isolated singular point on the boundary $\rho = 0$, which we may take to be the origin $\rho = z = 0$. Equivalently, in the Gibbons–Hawking metric, the horizon is an isolated singular point on the $z-$axis, which we may take to be the origin of $\mathbb{R}^3$. Furthermore, the harmonic functions can be written as*

$$H = \frac{h_0}{r} + H_0 \,, \qquad K = \frac{k_0}{r} + K_0 \,, \qquad L = \frac{\ell_0}{r} + L_0 \,, \qquad M = \frac{m_0}{r} + M_0 \,, \tag{2.97}$$

where $r = \sqrt{\rho^2 + z^2}$, H_0, K_0, L_0, M_0 are harmonic functions which are smooth at $r = 0$ and h_0, k_0, ℓ_0, m_0 are constants, where $h_0 \neq 0$ for a locally S^3 horizon and $h_0 = 0$ for a $S^1 \times S^2$ horizon. In addition, the parameters satisfy

$$-h_0^2 m_0^2 - 3h_0 k_0 \ell_0 m_0 + h_0 \ell_0^3 - 2k_0^3 m_0 + \frac{3}{4} k_0^2 \ell_0^2 > 0 \,, \tag{2.98}$$

which in particular also implies that $h_0 \ell_0 + k_0^2 > 0$.

Theorem 2.3. *Consider an asymptotically flat, supersymmetric and biaxisymmetric black hole solution to five-dimensional minimal supergravity.*

1. *Cross-sections of any connected component of the horizon must be homeomorphic to S^3, $S^1 \times S^2$ or a lens space $L(p, 1)$.*

2. *The coefficient of the singular term in the harmonic function H is $h_0 = \pm 1$ for an S^3 horizon, $h_0 = 0$ for $S^1 \times S^2$ and more generally $h_0 = p \in \mathbb{Z}$ for $L(p, 1)$.*

Remarks

1. Theorem 2.2 is a five-dimensional analogue of Theorem 3.2 in [28] which is a crucial ingredient for the classification of supersymmetric four-dimensional black hole spacetimes.

2. The horizon is an isolated singular point of the orbit space metric also for extremal vacuum solutions, see [45].

3. We will offer an alternative proof of part 1 of Theorem 2.3 by analysing the rod structure of the general solution, see section 2.5.1.

2.5 Geometry and topology of the axes

2.5.1 Rod structure

We now analyse the axes $\mathcal{A}$ in more detail. Recall that the axes are the part of the boundary of the orbit space $\hat{\mathcal{M}}$ where $\det \gamma_{ij} = 0$, i.e. the $U(1)^2$-symmetry has fixed points. In Weyl coordinates (ρ, z) the boundary $\partial\hat{\mathcal{M}}$ is the z-axis. As shown in the previous section, a horizon corresponds to an isolated singular point on the z-axis. The remaining part of the z-axis corresponds to $\mathcal{A}$, which splits into intervals along which γ_{ij} is of rank 1 with endpoints on which γ_{ij} is of rank 0. The intervals where γ_{ij} is rank-1 correspond to the axis rods, and the endpoints where $\gamma_{ij} = 0$ to the corners of the orbit space $\hat{\mathcal{M}}$.

Lemma 2.4. *Smoothness of the spacetime near an axis rod I implies $\hat{\omega} = \mathcal{O}(\rho^2)$ and $\chi = \chi|_I + \mathcal{O}(\rho^2)$ where $\chi|_I$ is an odd integer. The Killing field which vanishes on I is*

$$v = \partial_\phi - \chi|_I \partial_\psi \,, \tag{2.99}$$

where the normalisation has been fixed so the orbits of v (away from I) are 2π-periodic.

Proof. Note that we may write $\hat{\omega}$, χ in terms of spacetime invariants as

$$\hat{\omega} = \frac{1}{N} \begin{vmatrix} g_{t\psi} & g_{t\phi} \\ g_{\psi\psi} & g_{\psi\phi} \end{vmatrix} \,, \qquad \chi = -\frac{1}{N} \begin{vmatrix} g_{tt} & g_{t\psi} \\ g_{t\phi} & g_{\psi\phi} \end{vmatrix} \,. \tag{2.100}$$

Thus they are smooth functions wherever $N > 0$. In particular, as by Lemma 2.3 $N > 0$ has to be satisfied everywhere except at corners of the orbit space, this implies $\hat{\omega}$ and χ are smooth functions on the axis rods.

Let us study the metric near an axis rod I. We can expand the vector v that vanishes on I in terms of the vectors $v_\pm$ vanishing on the axis rods that stretch out to infinity, $v = av_+ + bv_-$, where a, b are coprime integers which are not both vanishing. By (2.24) this implies $v = (b-a)\partial_\psi + (a+b)\partial_\phi$. Since we know $\partial_\psi = 0$ can only occur at a corner of the orbit space (recall from (2.8) $\partial_\psi = 0$ implies $N = 0$), we must have $a + b \neq 0$. Hence on the axis rod we can write $\partial_\phi = c\partial_\psi$ with $c = (a-b)/(a+b)$. Then from (2.100), on the axis interval $\hat{\omega} = 0$ and $\chi = c$.

To determine the behaviour of $\hat{\omega}$ and χ near I we argue as follows. First, from (2.27), equation (1.92) is equivalent to $\partial_z \chi = -\rho\partial_\rho H$ and $\partial_\rho \chi = \rho\partial_z H$. We know the axisymmetric harmonic function H is smooth at I, so near I we can write $H = H_0(z) + \mathcal{O}(\rho^2)$ for some smooth function $H_0(z)$. Therefore, integrating it follows that $\chi = \chi|_I + \mathcal{O}(\rho^2)$ where $\chi|_I$ is a constant. From the above we deduce that $\chi|_I = c$. Similarly, integrating equation (1.96) for $\hat{\omega}$ and using the fact that the harmonic functions K, L, M are also smooth at I, implies $\hat{\omega} = \hat{\omega}|_I + \mathcal{O}(\rho^2)$ where $\hat{\omega}|_I$ is a constant. We have already seen that $\hat{\omega}$ must vanish on I, so the constant $\hat{\omega}|_I = 0$.

Putting things together we find the spacetime metric for $z \in I$ and $\rho \to 0$ is

$$\mathrm{d}s^2|_{\text{near } I} = -f^2\big(\mathrm{d}t + \mathcal{O}(\rho^2)\,\mathrm{d}\phi_I\big)^2 + \frac{H}{f}\big(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi_I^2 + \mathrm{d}z^2\big) + g_{\psi\psi}\big(\mathrm{d}\psi_I + \mathcal{O}(\rho^2)\,\mathrm{d}\phi_I\big)^2 \\ + 2g_{t\psi}\big(\mathrm{d}t + \mathcal{O}(\rho^2)\,\mathrm{d}\phi_I\big)\big(\mathrm{d}\psi_I + \mathcal{O}(\rho^2)\,\mathrm{d}\phi_I\big)\,, \quad (2.101)$$

where we have defined new coordinates $(\psi_I, \phi_I) = (\psi + \chi|_I\phi, \phi)$. Now, using smoothness of the harmonic functions, (2.1, 2.8) implies the invariants f, $g_{\psi\psi}$, $g_{\psi t}$, N must also be smooth on I, and near I the corrections must be $\mathcal{O}(\rho^2)$. Furthermore, by (2.9) and Lemma 2.3, $H/f > 0$ and $g_{\psi\psi} > 0$ on I. Hence, the above is a smooth Lorentzian metric iff ϕ_I is identified as $\phi_I \sim \phi_I + 2\pi$ (this can be seen by converting to Cartesian coordinates in the (ρ, ϕ_I) plane).

The above coordinate change implies $\partial_{\psi_I} = \partial_\psi$, $\partial_{\phi_I} = \partial_\phi - \chi_I\partial_\psi$. Thus the Killing vector v vanishing on I is $v = (a+b)(\partial_\phi - c\partial_\psi) = (a+b)\partial_{\phi_I}$. By our chosen normalisation, v must be periodic with period 2π, so since ϕ_I has period 2π we must have $a + b = 1$. Thus $v = (a+b)(\partial_\phi - \chi_I\partial_\psi)$ with $\chi|_I = c = (a-b)/(a+b) = 2a - 1$ indeed being an odd integer, as claimed, thereby completing the proof of the lemma. □

Let us now denote the axis rods by $I_i = (z_i, z_{i+1})$ for $i = 1, \dots, n-1$, where $z_1 < z_2 < \dots < z_n$, and $I_- = (-\infty, z_1)$ and $I_+ = (z_n, \infty)$. As we have just established, the 2π-normalised Killing fields vanishing on the respective axis rods are

$$v_i = \partial_\phi - \chi_i\partial_\psi\,, \qquad v_\pm = \partial_\phi - \chi_\pm\partial_\psi\,, \tag{2.102}$$

where $\chi_i \equiv \chi|_{I_i}$. The data $\{(I_i, v_i) \mid i, j = +, -, 1, .., n-1\}$ defines the rod structure of the spacetime [76].

There are certain compatibility requirements between adjacent rods that have been derived for stationary and biaxisymmetric spacetimes [76] (cf. also the preliminary section 1.1.1), which put the following constraints on the rod vectors: If v_i and v_j are the 2π-normalised rod vectors of adjacent axis rods, $\det(v_i^T v_j^T) = \pm 1$. If two axis rods, with vectors v_i and v_j, are separated only by a horizon, then $\det(v_i^T v_j^T) = p \in \mathbb{Z}$ and the topology of the horizon is $S^1 \times S^2$ for $p = 0$, S^3 for $p = \pm 1$, and in general a lens space $L(p, q)$ where $q \in \mathbb{Z}$ is only defined modulo p.

For asymptotically flat solutions $\chi_\pm = \pm 1$ and so $v_\pm$ coincide with (2.24), thus defining a natural 2π-normalised basis. In this basis the rod vectors (2.102) are

$$v_- = (1, 0)\,, \qquad v_i = (1 - a_i, a_i)\,, \qquad v_+ = (0, 1)\,, \tag{2.103}$$

where by Lemma 2.4 we have defined $a_i \equiv \frac{1}{2}(1 + \chi_i) \in \mathbb{Z}$ for $i = 1, \dots, n-1$. The determinants of adjacent rod vectors are then given by $(i = 1, \dots, n-1)$

$$\det(v_-^T\ v_1^T) = a_1\,, \qquad \det(v_i^T\ v_{i+1}^T) = a_{i+1} - a_i\,, \qquad \det(v_n^T\ v_+^T) = 1 - a_n\,. \tag{2.104}$$

Evidently, the rod structure is somewhat restricted. In particular, this provides extra constraints on the horizon topology, thus providing an alternative proof of Theorem 2.3 (part 1) as follows.

Proof of Theorem 2.3, part 1. Let v_{i-1}, v_i be the 2π normalised rod vectors of two rods separated by a horizon at $z = z_i$.[6] Following [76], we can pick an adapted basis (v_j, w_j) on a rod I_j $(j = +, -, 1, \dots, n-1)$ related to (v_+, v_-) by an $SL(2, \mathbb{Z})$ transformation (no summation over j),

$$\begin{pmatrix} v_j \\ w_j \end{pmatrix} = \begin{pmatrix} 1 - a_j & a_j \\ c_{(j)}(1 - a_j) - 1 & 1 + a_j c_{(j)} \end{pmatrix} \begin{pmatrix} v_- \\ v_+ \end{pmatrix} . \tag{2.105}$$

This defines w_j up to the integer $c_{(j)}$. Note that we have defined $a_- \equiv 0$, $a_+ \equiv 1$. Then for the two rods I_{i-1}, I_i, we may relate the two respective bases as

$$\begin{pmatrix} v_{i-1} \\ w_{i-1} \end{pmatrix} = \begin{pmatrix} 1 - c_{(i)}(a_{i-1} - a_i) & a_{i-1} - a_i \\ c_{(i-1)} - c_{(i)} - c_{(i-1)}c_{(i)}(a_{i-1} - a_i) & 1 - c_{(i-1)}(a_{i-1} - a_i) \end{pmatrix} \begin{pmatrix} v_i \\ w_i \end{pmatrix} . \tag{2.106}$$

The horizon topology is then determined by the coefficients (p, q) in

$$v_{i-1} = q v_i + p w_i \, . \tag{2.107}$$

Comparison with (2.106) gives

$$p = a_{i-1} - a_i \tag{2.108}$$

$$q = 1 - c_{(i)}(a_{i-1} - a_i) = 1 - c_{(i)} p \, , \tag{2.109}$$

so in particular since $c_{(i)}$ is an integer,

$$q = 1 \mod p \, . \tag{2.110}$$

This shows the horizon topology is $L(p, 1)$ as claimed. If $p = \pm 1$ this is just S^3, and if $p = 0$ this is $S^1 \times S^2$. □

2.5.2 Geometry of the axes

We now turn to the analysis of the metric on the axes. Consider an axis rod $I_i = (z_i, z_{i+1})$ where $z = z_i$ is a corner of the orbit space. The induced metric on I_i is

$$\mathrm{d}s^2|_{I_i} = -f^2 \,\mathrm{d}t^2 + \frac{H}{f}\,\mathrm{d}z^2 + g_{\psi\psi}\,\mathrm{d}\psi_i^2 + 2g_{t\psi}\,\mathrm{d}t\,\mathrm{d}\psi_i \, , \tag{2.111}$$

where $\psi_i = \psi + \chi_i \phi$. This is a 3-dimensional timelike submanifold with a circle action generated by $\partial_\psi = \partial_{\psi_i}$ which has a fixed point at the endpoint $z = z_i$. At this fixed

[6] Here the index $i = +, 1, \dots, n-1$ and we identify $(v_0, I_0) \equiv (v_-, I_-)$, $(v_n, I_n) \equiv (v_+, I_+)$.

point we must have $g_{\psi\psi} = g_{t\psi} = 0$. We will now analyse the conditions required by smoothness of the geometry near such a fixed point $z = z_i$.

It is convenient to use as a coordinate the proper distance from the fixed point,

$$s = \int_{z_i}^{z} \sqrt{g_{zz}}\, \mathrm{d}z\,. \tag{2.112}$$

Now, smoothness at $z = z_i$ requires that $4(\mathrm{d}|\partial_{\psi_i}|)^2 \to 1$ as $z \to z_i^+$ (recall that $\Delta\psi_i = 4\pi$). In terms of the proper distance this is equivalent to

$$g_{\psi\psi} = \tfrac{1}{4}s^2(1 + \mathcal{O}(s^2))\,, \tag{2.113}$$

where the subleading terms are fixed by smoothness at $s = 0$ (and converting to Cartesian coordinates in the (s, ψ_i)-plane). Smoothness of the metric and its inverse on the axis thus also requires that

$$g_{t\psi} = \mathcal{O}(s^2), \qquad f^2 = f_i^2 + \mathcal{O}(s^2)\,, \tag{2.114}$$

where $f_i = f|_{z=z_i} \neq 0$ by Lemma 2.3. This behaviour of the metric near $s = 0$, together with (2.9), gives $f/H = \frac{1}{4}f_i^2 s^2 + \mathcal{O}(s^4)$, hence using $g_{zz} = H/f$ we find

$$z - z_i = \int_0^s \sqrt{\frac{f}{H}}\, \mathrm{d}s = \tfrac{1}{4}|f_i|s^2(1 + \mathcal{O}(s^2))\,. \tag{2.115}$$

We deduce that

$$H = \frac{\mathrm{sgn}(f_i)}{z - z_i} + \mathcal{O}(1) \tag{2.116}$$

as $z \to z_i^+$. But, by Lemma 2.3, H is a harmonic function on $\mathbb{R}^3$ with an isolated singularity at $(\rho, z) = (0, z_i)$. Thus, (2.116) implies that the singularity of H is a pole of order one. Therefore, for $\rho \geq 0$, we must have

$$H = \frac{h_i}{\sqrt{\rho^2 + (z - z_i)^2}} + \tilde{H}_i\,, \tag{2.117}$$

where $h_i = \mathrm{sgn}(f_i)$ and $\tilde{H}_i$ is a harmonic function on $\mathbb{R}^3$ which is smooth at $(\rho, z) = (0, z_i)$.

We will now determine the behaviour of the other harmonic functions. Since $f_i \neq 0$, equation (2.11) implies that KH^{-1} is also smooth at any corner. We deduce that

$$K = \frac{k_i}{\sqrt{\rho^2 + (z - z_i)^2}} + \tilde{K}_i\,, \tag{2.118}$$

where k_i is a constant (possibly vanishing) and $\tilde{K}_i$ is a harmonic function smooth at the corner $(\rho, z) = (0, z_i)$. Next, smoothness of f^{-1} at the corner and (1.94) then implies

L may also have a pole of order one at the corner, so

$$L = \frac{\ell_i}{\sqrt{\rho^2 + (z - z_i)^2}} + \tilde{L}_i \,, \tag{2.119}$$

where $\ell_i = -h_i^{-1} k_i^2$ and $\tilde{L}_i$ is a harmonic function smooth at the corner. Finally, the invariant $-f^{-2} V \cdot \partial_\psi = \omega_\psi$ must be smooth at any fixed point of ∂_ψ (since $f_i \neq 0$), and thus (1.95) implies

$$M = \frac{m_i}{\sqrt{\rho^2 + (z - z_i)^2}} + \tilde{M}_i \,, \tag{2.120}$$

where

$$m_i = -k_i^3 - \tfrac{3}{2} h_i^{-1} k_i \ell_i = \tfrac{1}{2} k_i^3 \tag{2.121}$$

and $\tilde{M}_i$ is a harmonic function smooth at the corner. There are further conditions arising from the fact ω_ψ must also vanish at the corner which we will explore in more detail below.

Therefore, we have found that the boundary conditions arising from smoothness of the solution on the axes are sufficient to determine its functional form near any corner of the orbit space. The obtained conditions on the solution are *necessary* conditions for smoothness at a corner of the orbit space. In the following section we will show that in fact they are also sufficient.

2.5.3 Smoothness at corners of orbit space

In this section we complete the smoothness analysis at the corners of the orbit space. To do so, let us introduce $\mathbb{R}^3$-polar coordinates (r, θ, ϕ) centred at the corner $(\rho, z) = (0, z_i)$, where for notational simplicity we have dropped the label i in the new coordinates. Then, as just shown, we can write

$$H = \frac{h}{r} + \tilde{H} \,, \qquad K = \frac{k}{r} + \tilde{K} \,, \qquad L = \frac{\ell}{r} + \tilde{L} \,, \qquad M = \frac{m}{r} + \tilde{M} \,, \tag{2.122}$$

where $h = \pm 1$, $\ell = -h^{-1} k^2$, $m = k^3/2$ and $\tilde{H}$, $\tilde{K}$, $\tilde{L}$, $\tilde{M}$ are axisymmetric harmonic functions that are smooth at the centre. Thus we can write

$$\tilde{H} = \sum_{l=0}^{\infty} h_l r^l P_l(\cos\theta) \tag{2.123}$$

where P_l are the Legendre polynomials and h_l are constants, and similarly for $\tilde{K}$, $\tilde{L}$, $\tilde{M}$, where furthermore the constants h_l, k_l, ℓ_l, m_l are such that $\omega_\psi|_{r=0} = 0$. We will now show that for asymptotically flat solutions this is sufficient for smoothness at $r = 0$, assuming that on the adjacent rods χ and $\hat{\omega}$ are as required by Lemma 2.4.

Given (2.122), (2.123) and (2.27), we may solve (1.92) for the 1-form χ, giving

$$\chi = (h \cos\theta + \chi_0) \, \mathrm{d}\phi + \tilde{\chi} \,, \tag{2.124}$$

where χ_0 is a constant and, using the fact that P_l is a Legendre polynomial,

$$\tilde{\chi} = r^2 \sin^2\theta \sum_{l=1}^{\infty} \frac{h_l r^{l-1}}{l+1} P_l'(\cos\theta)\, \mathrm{d}\phi\,. \tag{2.125}$$

Now, define new coordinates (R, ψ', ϕ') by

$$r = \tfrac{1}{4}R^2, \qquad \psi' = \psi + \chi_0\phi, \qquad \phi' = h\phi\,, \tag{2.126}$$

so that the Gibbons–Hawking base is

$$\mathrm{d}s^2_{GH} = F\left(\mathrm{d}R^2 + \tfrac{1}{4}R^2\left[(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{1}{F^2}(\mathrm{d}\psi' + \cos\theta\,\mathrm{d}\phi' + \tilde{\chi})^2\right]\right) \tag{2.127}$$

where we have defined $F \equiv rH = h + r\tilde{H}$. In terms of the new coordinates

$$F = \pm 1 + \mathcal{O}(R^2)\,, \qquad \tilde{\chi} = \mathcal{O}(R^4)\,\mathrm{d}\phi'\,, \tag{2.128}$$

so we see that as $R \to 0$, the Gibbons–Hawking base approaches the origin $\pm\mathbb{R}^4$, *if* the angles (ψ', ϕ') are identified as Euler angles on S^3. Since the original angles (ψ, ϕ) are required to be Euler angles on S^3 by asymptotic flatness, this implies the identifications

$$(\psi', \phi') \sim (\psi' + 4\pi, \phi') \sim (\psi' + (\chi_0 + 1)2\pi, \phi' + 2\pi)\,, \tag{2.129}$$

so (ψ', ϕ') are also Euler angles on S^3 if and only if $\chi_0 + 1 = 2n + 1$ for some $n \in \mathbb{Z}$, i.e. if and only if χ_0 is an even integer. In fact, this condition follows from Lemma 2.4: On the axis $\theta = 0, \pi$ it is clear that $\tilde{\chi} = 0$ and thus on any axis rod I we have $\chi|_I = \chi_0 \pm h$. As required by Lemma 2.4, we have fixed $\chi|_I$ to be an odd integer. Hence χ_0 is an even integer, as required.

In order to verify the Gibbons–Hawking base at the centre is actually smooth requires us to control the higher order terms more carefully. To this end introduce coordinates[7]

$$\phi^{\pm} = \tfrac{1}{2}(\psi' \pm \phi')\,, \qquad X_+ = R\cos(\tfrac{1}{2}\theta)\,, \qquad X_- = R\sin(\tfrac{1}{2}\theta)\,, \tag{2.130}$$

so

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}X_+^2 + X_+^2(\mathrm{d}\phi^+)^2 + \mathrm{d}X_-^2 + X_-^2(\mathrm{d}\phi^-)^2 \tag{2.131}$$

and $\phi^{\pm}$ are independently 2π periodic. In these coordinates any smooth biaxisymmetric function on $\mathbb{R}^4$ is a smooth function of (X_+^2, X_-^2). Noting that

$$r = \tfrac{1}{4}(X_+^2 + X_-^2)\,, \qquad r\cos\theta = \tfrac{1}{4}(X_+^2 - X_-^2)\,, \tag{2.132}$$

[7] The coordinates $\phi^{\pm}$ in this section are different to those in (2.24). We will only use these in this section, so there should be no confusion.

and using the fact that P_l are polynomials of order l, it is easy to see that $\tilde{H}$ and hence F are analytic functions of (X_+^2, X_-^2). Similarly we find that

$$\tilde{\chi} = \tfrac{1}{4} X_+^2 X_-^2 (h_1 + \dots) h \, \mathrm{d}\phi' = \tfrac{1}{4} X_+^2 X_-^2 (h_1 + \dots) h (\mathrm{d}\phi^+ - \mathrm{d}\phi^-) \,, \tag{2.133}$$

where the higher order terms are analytic in (X_+^2, X_-^2), so the 1-form $\tilde{\chi}$ is analytic at the origin of $\mathbb{R}^4$. Putting everything together, we can write the Gibbons–Hawking base as

$$\begin{aligned} \mathrm{d}s^2_{GH} = F \, \mathrm{d}s^2(\mathbb{R}^4) - \frac{\tilde{H}(h+F)}{4F} (X_+^2 \, \mathrm{d}\phi^+ + X_-^2 \, \mathrm{d}\phi^-)^2 \\ + \frac{1}{F} (X_+^2 \, \mathrm{d}\phi^+ + X_-^2 \, \mathrm{d}\phi^-) \tilde{\chi} + \frac{F(X_+^2 + X_-^2)}{4} \tilde{\chi}^2 \end{aligned} \tag{2.134}$$

which is now manifestly analytic at the origin of $\mathbb{R}^4$. Therefore, the Gibbons–Hawking base is indeed smooth, in fact analytic, at any centre corresponding to a corner of the orbit space.

We now turn to the other components of the spacetime metric, namely the function f and 1-form ω. Expanding the regular parts $\tilde{K}$, $\tilde{L}$, $\tilde{M}$ of the harmonic functions K, L, M as above for $\tilde{H}$ it is easy to see that f is an analytic function of (X_+^2, X_-^2). Recall by Lemma 2.3 we must have $f \neq 0$ at any centre corresponding to a corner of the orbit space.

It remains to be checked that also the 1-form ω is smooth at the centre. In the above coordinates we can write

$$\omega = \omega_\psi (\mathrm{d}\psi' + \cos\theta \, \mathrm{d}\phi') + \omega_\psi \tilde{\chi} + \hat{\omega} = \frac{2\omega_\psi}{R^2} (X_+^2 \, \mathrm{d}\phi^+ + X_-^2 \, \mathrm{d}\phi^-) + \omega_\psi \tilde{\chi} + \hat{\omega} \,. \tag{2.135}$$

Using (1.95) and expanding

$$\frac{1}{h + r\tilde{H}} = h - r\tilde{H} + r^2 G_1 \,, \tag{2.136}$$

where $G_1 = \tilde{H}^2/(h + r\tilde{H})$ is analytic in (X_+^2, X_-^2), as well as making use of the identities $h^2 = 1$, $\ell = -hk^2$, $m = k^3/2$, one finds

$$\omega_\psi = \sum_{l=0}^{\infty} \left(m_l - hm h_l + \frac{3}{2} (hk\ell_l - h\ell k_l) \right) r^l P_l(\cos\theta) + r\tilde{G}_1 \,, \tag{2.137}$$

where $\tilde{G}_1$ is some analytic function in (X_+^2, X_-^2). Thus ω_ψ, and hence also $\omega_\psi \tilde{\chi}$ are analytic in (X_+^2, X_-^2) and for smoothness of (2.135) we therefore only need to check that

$$\left(\frac{2\omega_\psi X_+^2}{R^2} + h\hat{\omega}_\phi \right) \mathrm{d}\phi^+ + \left(\frac{2\omega_\psi X_-^2}{R^2} - h\hat{\omega}_\phi \right) \mathrm{d}\phi^- \tag{2.138}$$

is smooth at the origin, or equivalently that

$$\frac{2\omega_\psi X_\pm^2}{R^2} \pm h\hat{\omega}_\phi = X_\pm^2 G_\pm \tag{2.139}$$

for some smooth functions $G_\pm$ of (X_+^2, X_-^2).

In fact one can solve equation (1.96) for the 1-form $\hat{\omega}$, of the form (2.27), as

$$\hat{\omega}_\phi = \hat{\omega}_0 + r\sin^2\theta \sum_{l=1}^{\infty} \frac{(hm_l - h_l m) + \frac{3}{2}(k\ell_l - k_l\ell)}{l} r^{l-1} P_l'(\cos\theta) + hr^2\sin^2\theta G_2\,, \tag{2.140}$$

where we have used that $\omega_\psi|_{r=0} = 0$ and defined

$$G_2 = h\sum_{j=0}^{\infty}\sum_{l=1}^{\infty} \frac{h_j m_l - h_l m_j + \frac{3}{2}(k_j\ell_l - k_l\ell_j)}{l+j+1} r^{l+j-1} P_l'(\cos\theta) P_j(\cos\theta) \tag{2.141}$$

which is an analytic function of (X_+^2, X_-^2). As required by Lemma 2.4, we have $\hat{\omega}_0 = 0$. Then

$$\begin{aligned}\frac{2\omega_\psi X_\pm^2}{R^2} \pm h\hat{\omega}_\phi = \frac{X_\pm^2}{2}\tilde{G}_1 \pm \frac{X_+^2 X_-^2}{4} G_2 + \\ h\sum_{l=1}^{\infty}\left(hm_l - mh_l + \frac{3}{2}(k\ell_l - \ell k_l)\right) r^l \left((1\pm\cos\theta)P_l(\cos\theta) \pm \frac{\sin^2\theta}{l}P_l'(\cos\theta)\right),\end{aligned} \tag{2.142}$$

and it is obvious that the first two terms are of the required form. Using basic properties of Legendre polynomials we can rewrite

$$(1\pm\cos\theta)P_l(\cos\theta) \pm \frac{\sin^2\theta}{l}P_l'(\cos\theta) = P_l(\cos\theta) \pm P_{l-1}(\cos\theta)\,. \tag{2.143}$$

Furthermore, from the recursion formula for Legendre polynomials, it follows that[8]

$$r^l[(P_l(\cos\theta) \pm P_{l-1}(\cos\theta)] = r(1\pm\cos\theta)\tilde{G}_\pm = \frac{X_\pm^2}{2}\tilde{G}_\pm \tag{2.144}$$

for some analytic $\tilde{G}_\pm$, so indeed (2.139) is satisfied. This establishes that the 1-form ω is smooth, in fact analytic, at any centre corresponding to a corner of the orbit space.

Putting things together, we have shown that the spacetime metric is analytic at any point corresponding to a corner of the orbit space. Furthermore, near such points the spacetime is diffeomorphic to $\mathbb{R}^{1,4}$.

The gauge field in the new coordinates takes the form

$$A = \frac{\sqrt{3}}{2}\left(f\,\mathrm{d}t + A_+\,\mathrm{d}\phi^+ + A_-\,\mathrm{d}\phi^-\right), \tag{2.145}$$

[8] This follows easily by induction from writing the recursion formula in the form $(l+1)(P_{l+1} \pm P_l) = \mp l(P_l \pm P_{l-1}) \pm (1\pm\cos\theta)(2l+1)P_l$ and noting that $P_1 \pm P_0 = \cos\theta \pm 1$.

where

$$A_{\pm} = f\Big(\frac{2\omega_\psi X_{\pm}^2}{R^2} \pm h\hat{\omega}_\phi\Big) \pm h\big(f\omega_\psi - \frac{K}{H}\big)\tilde{\chi}_\phi - \frac{2X_{\pm}^2}{R^2}\frac{K}{H} \mp h\xi_\phi \,. \tag{2.146}$$

Clearly A_t is analytic at $R = 0$. We have already shown (2.139), so the first term in (2.146) is analytic and proportional to $X_{\pm}^2$. As f, ω_ψ, K/H are analytic at the centre and $\tilde{\chi}$ is of the form (2.133), the same is true for the second term. Lastly, integrating (1.98), for ξ of the form (2.27), gives

$$\xi = \left(\xi_0 - k\cos\theta - \frac{X_+^2 X_-^2}{4}\sum_{l=1}^{\infty}\frac{k_l r^{l-1}}{l+1}P_l'(\cos\theta)\right)\mathrm{d}\phi\,, \tag{2.147}$$

and hence, using (2.136),

$$-\frac{2X_{\pm}^2}{R^2}\frac{K}{H} \mp h\xi_\phi = -hk \mp h\xi_0 + X_{\pm}^2(\dots) + X_+^2X_-^2(\dots)\,, \tag{2.148}$$

where $\dots$ are analytic functions of (X_+^2, X_-^2). Thus A is gauge-equivalent to an analytic 1-form. Therefore the Maxwell field (hence the full solution) is analytic at the centre.

Finally, we emphasise that the above analysis shows that the solution is smooth and locally stably causal at and near any centre corresponding to a corner of the orbit space. Indeed we have,

$$K^2 + HL = \frac{1}{|f|r} + \mathcal{O}(1), \qquad g^{tt} = -\frac{1}{f^2} + \mathcal{O}(r)\,, \tag{2.149}$$

where recall that $f \neq 0$ at the centre, thus confirming the solution is smooth (2.16) and causal (2.18) near the centre.

2.5.4 Summary

To summarise, we have shown the following.

Theorem 2.4. *Let $(\mathcal{M}, g, F)$ be an asymptotically flat, supersymmetric and biaxisymmetric solution to minimal supergravity with a globally hyperbolic domain of outer communication $\langle\langle\mathcal{M}\rangle\rangle$. Let $(\rho, z) = (0, z_i)$ be a point corresponding to a corner of the orbit space. Then the solution is smooth (in fact analytic) at the corner if and only if $f_i \equiv f|_{(\rho,z)=(0,z_i)} \neq 0$, $\omega_\psi|_{(\rho,z)=(0,z_i)} = 0$, on the adjacent rods $I \in \{I_{i-1}, I_i\}$ $\chi|_I$ are odd integers and $\hat{\omega}|_I = 0$, and the harmonic functions H, K, L, M are given by*

$$H = \frac{h_i}{r_i} + \tilde{H}_i\,, \qquad K = \frac{k_i}{r_i} + \tilde{K}_i\,, \qquad L = -\frac{h_i^{-1}k_i^2}{r_i} + \tilde{L}_i\,, \qquad M = \frac{\frac{1}{2}k_i^3}{r_i} + \tilde{M}_i\,, \tag{2.150}$$

where $r_i = \sqrt{\rho^2 + (z - z_i)^2}$, $h_i = sgn(f_i)$, k_i are constants and $\tilde{H}_i$, $\tilde{K}_i$, $\tilde{L}_i$, $\tilde{M}_i$ are harmonic functions on $\mathbb{R}^3$ which are smooth at $(\rho, z) = (0, z_i)$. Furthemore, the spacetime near such a corner is diffeomorphic to $\mathbb{R}^{1,4}$.

2.6 Classification theorem

We will now combine the constraints obtained from the existence of a smooth horizon in section 2.4 and smooth axes in section 2.5 and give our main classification theorem.

Theorem 2.5. *Consider an asymptotically flat, supersymmetric and biaxisymmetric solution to minimal supergravity with a smooth globally hyperbolic domain of outer communication and a smooth event horizon with compact cross-sections (if there is a black hole). Suppose the orbit space $\hat{\mathcal{M}}$ has k corners and the horizon has l connected components ($l = 0$ corresponds to no black hole), and let $n = k + l$. Then, the harmonic functions are*

$$H = \sum_{i=1}^{n} \frac{h_i}{r_i}\,, \qquad K = \sum_{i=1}^{n} \frac{k_i}{r_i}\,, \qquad L = 1 + \sum_{i=1}^{n} \frac{\ell_i}{r_i}\,, \qquad M = m + \sum_{i=1}^{n} \frac{m_i}{r_i}\,, \tag{2.151}$$

where $r_i = \sqrt{\rho^2 + (z - z_i)^2}$, $(\rho, z) = (0, z_i)$ are corners of the orbit space or horizons, and

$$\sum_{i=1}^{n} h_i = 1\,, \qquad m = -\tfrac{3}{2} \sum_{i=1}^{n} k_i\,. \tag{2.152}$$

The corresponding 1-forms can be written as

$$\begin{aligned} \chi &= \sum_{i=1}^{n} \frac{h_i(z - z_i)}{r_i}\, \mathrm{d}\phi\,, \qquad \xi = -\sum_{i=1}^{n} \frac{k_i(z - z_i)}{r_i}\, \mathrm{d}\phi\,, \\ \hat{\omega} &= \Bigg[-\sum_{i=1}^{n} \frac{(m h_i + \frac{3}{2} k_i)(z - z_i)}{r_i} \\ &\qquad - \sum_{i=1}^{n} \sum_{j \neq i} \left(\frac{h_i m_j + \frac{3}{2} k_i \ell_j}{z_i - z_j} \right) \left(\frac{\rho^2 + (z - z_i)(z - z_j)}{r_i r_j} - 1 \right) \Bigg] \mathrm{d}\phi\,, \end{aligned} \tag{2.153}$$

and the parameters have to satisfy for each $i = 1, \ldots, n$,

$$h_i m + \tfrac{3}{2} k_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{h_i m_j - m_i h_j - \frac{3}{2}(\ell_i k_j - k_i \ell_j)}{|z_i - z_j|} = 0\,. \tag{2.154}$$

Furthermore, if $(0, z_i)$ is a corner, $h_i = \pm 1$ and the parameters must satisfy

$$\ell_i = -h_i^{-1} k_i^2\,, \qquad m_i = \tfrac{1}{2} k_i^3\,, \tag{2.155}$$

$$h_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{2 k_i k_j - h_i (h_j k_i^2 - \ell_j)}{|z_i - z_j|} > 0\,. \tag{2.156}$$

On the other hand, if $(0, z_i)$ is a horizon the parameters must satisfy $h_i \in \mathbb{Z}$,

$$-h_i^2 m_i^2 - 3 h_i k_i \ell_i m_i + h_i \ell_i^3 - 2 k_i^3 m_i + \frac{3}{4} k_i^2 \ell_i^2 > 0\,, \tag{2.157}$$

(which also implies $h_i\ell_i + k_i^2 > 0$) and cross-sections of the horizon are of topology S^3 if $h_i = \pm 1$, $S^2 \times S^1$ if $h_i = 0$ and the lens space $L(h_i, 1)$ otherwise.

Proof. We have shown in Theorem 2.2 that a horizon corresponds to at most a simple pole of the harmonic functions H, K, L, M. Similarly, a corner corresponds to a simple pole of H and at most a simple pole of K, L, M, see Theorem 2.4. Hence, with the stated assumptions we can write

$$H = \tilde{H} + \sum_{i=1}^{n} \frac{h_i}{r_i}\,, \qquad K = \tilde{K} + \sum_{i=1}^{n} \frac{k_i}{r_i}\,, \qquad L = \tilde{L} + \sum_{i=1}^{n} \frac{\ell_i}{r_i}\,, \qquad M = \tilde{M} + \sum_{i=1}^{n} \frac{m_i}{r_i}\,, \tag{2.158}$$

where $\tilde{H}$, $\tilde{K}$, $\tilde{L}$, $\tilde{M}$ are harmonic functions that are smooth at $(\rho, z) = (0, z_i)$ for all $i = 1, \ldots, n$. By Lemma 2.3, the only singularities of H, K, L, M in the DOC are at points corresponding to the corners of the orbit space. Therefore, $\tilde{H}$, $\tilde{K}$, $\tilde{L}$, $\tilde{M}$ must be smooth on all of $\mathbb{R}^3$. Asymptotic flatness (2.25) implies these harmonic functions are bounded. Therefore, $\tilde{H}$, $\tilde{K}$, $\tilde{L}$, $\tilde{M}$ are smooth and bounded harmonic functions on $\mathbb{R}^3$, hence they must be constants which coincide with their asymptotic values, so $\tilde{H} = 0$, $\tilde{L} = 1$, $\tilde{K} = 0$ and $\tilde{M} = m$. The asymptotic flatness conditions (2.25) and (2.26) then reduce to (2.152). This establishes the form of the harmonic functions.

Given the harmonic functions, the 1-forms are easily obtained by integration from (1.92, 1.96, 1.98). The integration constants in χ and $\hat{\omega}$ have been fixed so that $\chi = \cos\theta + \mathcal{O}(r^{-1})$ and $\hat{\omega} = \mathcal{O}(r^{-1})$ as $r \to \infty$, as required by asymptotic flatness.

The constraints on the parameters at each corner (2.155) are given in Theorem 2.4. The additional constraint (2.156) is equivalent to the condition $h_i f_i > 0$, which also follows from Theorem 2.4. The constraints on the parameters at a horizon (2.157) are given in Theorems 2.2 and 2.3.

The constraints (2.154) are equivalent to $\hat{\omega} = 0$ on each of the axis rods $I_i = (z_i, z_{i+1})$, which is required by smoothness at the axes, see Lemma 2.4 and Theorem 2.4. This can be seen as follows. From (2.153) it is obvious that $\hat{\omega}$ is constant on any axis rod I_i. Furthermore on the semi-infinite axis rods $I_\pm$,

$$\hat{\omega}|_{I_\pm} = \mp \sum_{i=1}^{n} (h_i m + \tfrac{3}{2} k_i) = 0 \tag{2.159}$$

by the asymptotic flatness conditions (2.152). Thus $\hat{\omega}$ vanishes on all axis rods if and only if the difference $\hat{\omega}|_{I_i} - \hat{\omega}|_{I_{i-1}}$ is zero for all $i = 1, \ldots, n$ (we identified $I_0 \equiv I_-$). Computing yields that the difference between $\hat{\omega}$ evaluated on two adjacent rods separated by the centre $(0, z_i)$ is given by

$$\hat{\omega}|_{I_i} - \hat{\omega}|_{I_{i-1}} = -2 \left(h_i m + \tfrac{3}{2} k_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{h_i m_j - m_i h_j - \frac{3}{2}(\ell_i k_j - k_i \ell_j)}{|z_i - z_j|} \right) \tag{2.160}$$

which is (up to a factor -2) just the lefthand side of (2.154). Therefore indeed $\hat{\omega}|_{I_i} = 0$ for all $i = 1, \dots n-1$ precisely if (2.154) is satisfied. It is worth noting that for any corner $(0, z_i)$ the condition (2.154) is in fact equivalent to $\omega_\psi|_{r_i=0} = 0$, as is also required by Theorem 2.4.

Finally, we note that the other condition required for smooth axes, given in Lemma 2.4, is that χ evaluated on each axis rod has to be an odd integer. Evaluating (2.153) on each axis rod, we find this is automatically satisfied since h_i are integers for all $i = 1, \dots, n$ (see equation (3.9)).

□

Remarks

1. This shows that supersymmetric black holes and solitons, *must* be multi-centred solutions with a Gibbons–Hawking base. This is a five-dimensional analogue of Corollary 4.2 in [28].

2. It shall be stressed that while (2.151)–(2.157) are sufficient for smoothness at the horizon and the centres, to confirm that the solution is smooth and stably causal everywhere in the DOC one must check the condition (2.16) in Lemma 2.1, and (2.18). Our analysis shows that these are indeed satisfied near infinity, near the horizon and near any point corresponding to a corner of the orbit space. We know by now (see chapter 4 of this thesis) that in general (2.16) will indeed put further constraints on the parameters. We will discuss this in more detail for 3-centred solutions in section 4.2.

Chapter 3

Moduli space of soliton and black hole solutions in five dimensions

3.1 Introduction

We have shown in the previous chapter that any asymptotically flat, supersymmetric and biaxisymmetric solution to minimal supergravity in five dimensions must be of the form laid out in Theorem 2.5. Some solutions in this class are known. This includes the BMPV black hole [15], the supersymmetric black ring [37], black lenses [86, 112] or the "black hole with bubble" solution [85], but also the soliton solutions [11]. This chapter gives a detailed analysis of the moduli space $\mathcal{M}$ of solutions defined by Theorem 2.5.[1]

The chapter is organised as follows. After discussing some general properties of the moduli space in section 3.2 we will look at specific solutions in section 3.3. We will recover the aforementioned known solutions, but also find (in general infinitely many) solutions that have not been discussed in the literature before. Finally we discuss the physical properties of a general solution in section 3.4.

The results presented in this chapter have first been published in [16].

3.2 Structure of the moduli space

We can see from Theorem 2.5 that a general solution with $n = k + l$ centres will be determined by the (discrete) n-dimensional vector $h = (h_1, \ldots, h_n)$, as well as $(4n - 1)$ real parameters,

$$\Big\{\{z_{i+1} - z_i\}_{i=1,\ldots,n-1},\, \{k_i, \ell_i, m_i\}_{i=1,\ldots,n}\Big\}, \tag{3.1}$$

subject to $3k + l$ constraint equations (2.154)–(2.155), of which the $2k$ constraints (2.155) can be solved algebraically. Furthermore, there is a remaining one-parameter

[1] Note that we use the same symbol for the moduli space as was used for the spacetime manifold in previous chapters. The meaning should be clear from context.

gauge freedom (2.14)–(2.15) in the harmonic functions under which

$$k_i \to k_i + ch_i\,, \qquad \ell_i \to \ell_i - 2ck_i - 2c^2 h_i\,, \qquad m_i \to m_i - \tfrac{3}{2}c\ell_i + \tfrac{3}{2}c^2 k_i + \tfrac{1}{2}c^3 h_i\,. \tag{3.2}$$

Summing up, we find that the moduli space of $(k+l)$-centred solutions, $\mathcal{M}^{k,l}$, is given by the subset of the $(2k+4l-1)$-dimensional parameter space

$$\Big\{\{z_{i+1}-z_i\}_{i=1,\ldots,n-1},\, \{k_i\}_{i=1,\ldots,n},\, \{\ell_j, m_j\}_{\text{if } z_j \text{ is a horizon}}\Big\}\,, \tag{3.3}$$

defined by the set of $k+l$ polynomial equations (2.154) subject to the inequalities (2.156) and equivalence relations (3.2). By a general count of degrees of freedom,

$$\dim \mathcal{M}^{k,l} = k + 3l - 2 - \tilde{\Delta} + \Delta(k,l)\,, \tag{3.4}$$

where $\tilde{\Delta}$ has been introduced to correct for any potential restrictions on the parameters coming from (2.16, 2.18) (see Remark 2 at the end of the previous chapter), and the second correction term $\Delta(k,l)$ to accomodate for a potential redundancy in equations (2.154). One can easily see that there is at least one such redundancy, as summing (2.154) over all i gives

$$m\sum_{i=1}^{n} h_i + \frac{3}{2}\sum_{i=1}^{n} k_i + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_i m_j - m_i h_j - \frac{3}{2}(\ell_i k_j - k_i \ell_j)}{|z_i - z_j|} = 0\,, \tag{3.5}$$

where we have made use of (2.152) and (2.155), and the double sum vanishes for reasons of symmetry. We thus know that

$$1 \le \Delta(k,l) \le k + l. \tag{3.6}$$

In fact on the basis of known examples, we will conjecture that $\tilde{\Delta} = 0$,[2] $\Delta(k,l) = 1$, so

$$\dim \mathcal{M}^{k,l} = k + 3l - 1\,. \tag{3.7}$$

Indeed, this agrees with the known solutions which are discussed below.

When counting the number of solutions for a given k, l, it is important to realise there is a redundancy in our parameterisation corresponding to a discrete global isometry,

$$z \to -z\,, \qquad z_i \to -z_{n-i+1}\,, \qquad \phi \to -\phi\,, \qquad i \to n - i + 1\,. \tag{3.8}$$

Now, each separate choice of h will define a component of the moduli space. The number of connected components of $\mathcal{M}^{k,l}$ is thus given by the number of possible choices of h, taking into account the remaining reflection symmetry (3.8) of the axis.[3] As we

[2] Even in the known cases where (2.16, 2.18) do add constraints on the parameter space, see Remark 2 at the end of the previous chapter, these do not affect its dimension.

[3] In general the constraints (2.154–2.157, 2.16, 2.18) could lead to disconnected components of the

will show next, the choice of h is precisely equivalent to a choice of rod structure of the solution, so the number of components of the moduli space is also the number of inequivalent rod structures compatible with (2.151)–(2.157).

As we have seen earlier, the centres $z = z_i$ split the z-axis into $n+1$ intervals, $I_\pm, I_i$, on each of which the respective Killing field (2.102) vanishes. Having the full solution at hand, we can now explicitly evaluate χ on each of these intervals as

$$\chi_\pm = \pm 1, \qquad \chi_i \equiv \chi|_{I_i} = \sum_{j=1}^{i} h_j - \sum_{j=i+1}^{n} h_j = 2\sum_{j=1}^{i} h_j - 1\,, \tag{3.9}$$

where the final equality follows from the asymptotic condition (2.152). Therefore, in the basis defined by (2.24), one finds the rod vectors are given by (2.103) with

$$a_i = \sum_{j=1}^{i} h_j\,. \tag{3.10}$$

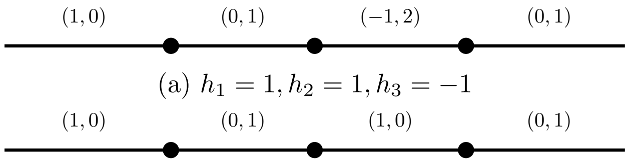
The determinants of adjacent rod vectors (2.104) are then precisely given by the value of h_i at the respective centre,

$$\det(v_-^T\ v_1^T) = h_1\,, \qquad \det(v_i^T\ v_{i+1}^T) = h_{i+1}\,, \qquad \det(v_n^T\ v_+^T) = h_n\,. \tag{3.11}$$

Indeed the conditions from our horizon and axes analysis, which say that if z_i is a horizon, $h_i = p$ is the parameter determining the horizon topology ($S^1 \times S^2$ if $p = 0$, S^3 if $p = \pm 1$ or in general $L(p, 1)$), and if z_i is a corner, $h_i = \pm 1$, are in precise agreement with the compatibility conditions for adjacent rod vectors previously derived for stationary and biaxisymmetric spacetimes [76]. Therefore, these compatibility conditions are automatically satisfied for solutions of the form (2.151)–(2.157) and impose no extra constraints.

The topology of the domain of outer communication is nontrivial and determined by the rod structure. The internal axis rods I_i ($i = 1, \ldots, n-1$), or indeed any simple curve in the $\mathbb{R}^3$ base of the Gibbons–Hawking space between the endpoints of I_i, together with the $U(1)$-fibre generated by ∂_ψ over the base, correspond to noncontractible 2-cycles C_i. The topology of the 2-cycle depends on the type of endpoint: If the endpoints of I_i are both corners of the orbit space, then the ψ-fibre collapses smoothly at the endpoints, so C_i is a surface of S^2 topology. If one endpoint of I_i is a corner and one a horizon, then ∂_ψ only vanishes at one of the endpoints, so C_i is a surface of 2-disc topology, with the boundary of the disc attached to the horizon. Finally, if both endpoints are horizons then C_i is a 2-tube extending between the two horizons.

moduli space even for fixed h. Thus strictly speaking the number of possible choices of h gives a *lower bound* on the number of connected components of the moduli space.



(b) $h_1 = 1, h_2 = -1, h_3 = 1$

Figure 3.1: Inequivalent rod structures for three-centred solitons. Note that rod lengths are encoded in the parameters $\{z_i\}$ of the solution and not specified in this depiction.

3.3 Solutions

3.3.1 Soliton solutions

Let us first consider the moduli space of n-centred soliton solutions, $\mathcal{M}^{n,0}$. Since every centre corresponds to a corner of the orbit space we must have $h_i = \pm 1$ for all $i = 1, \dots n$. On the other hand, asymptotic flatness requires $\sum_{i=1}^{n} h_i = 1$. It follows that soliton solutions will necessarily have an odd number of centres, $n = 2m+1$, where m is the number of $h_i = -1$. We can now easily determine the number of distinct rod structures this allows for. There are $\binom{n}{m}$ possible ways of choosing $h \equiv (h_1, \dots, h_n)$. Some of these, however, will be related by the discrete reflection symmetry (3.8) (which implies $h_i \to h_{n-i+1}$) and thus correspond to isometric solutions. Correcting for this overcounting, one finds the number of connected components of the moduli space to be given by

$$N(\mathcal{M}^{n,0}) = \frac{1}{2}\left[\binom{n}{m} + \binom{m}{[m/2]}\right] , \tag{3.12}$$

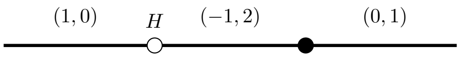
where the latter term arises as a correction for solutions which are themselves symmetric under reflection (and thus had *not* falsely been overcounted before).

For $n = 1$, the only possible solution without a black hole is Minkowski space. The allowed inequivalent rod structures for $n = 3$ are defined by $h = (1, 1, -1)$ and $h = (1, -1, 1)$ and are depicted in Figure 3.1. We see that there are two inequivalent soliton solutions in this case. The above counting formula shows that the number of inequivalent soliton solutions increases with n.

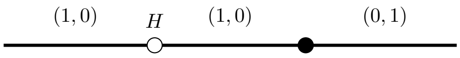
The n-centred soliton solutions correspond to asymptotically flat, globally hyperbolic regular spacetimes containing $n - 1$ noncontractible 2-cycles, or "bubbles". Such bubbling spacetimes were first constructed in [11] and some of their global properties elucidated in [56].

3.3.2 Single black hole solutions

We now consider the moduli space for n-centred solutions with a single black hole, $\mathcal{M}^{n-1,1}$. Thus, for one centre, say z_j, the determinant of the matrix of adjacent rod vectors $h_j = p \in \mathbb{Z}$ while the other centres correspond to corners, so $h_i = \pm 1$ for all



(a) $p = h_1 = 2$, $h_2 - 1$



(b) $p = h_1 = 0$, $h_2 = 1$

Figure 3.2: Rod structures for two-centred single black hole solutions.

$i \neq j$. As we have seen, this means that the centre $z = z_j$ corresponds to a horizon of topology $L(p,1)$. Let us denote the number of corners with $h_i = \pm 1$ by $n_\pm$ so $n_+ + n_- + 1 = n$. Asymptotic flatness (2.152) also implies that $n_+ - n_- + p = 1$. It follows that

$$p = n - 2n_+ \tag{3.13}$$

where $0 \leq n_+ \leq n-1$. Hence p is even for an even number of centres and odd otherwise, and the possible values of p are $-n+2, -n+4, \ldots, n-2, n$.

For a given p, there are $n\binom{n-1}{n_+}$ ways of choosing h (the factor n comes from different positions of the centre). However, some of these configurations will be related by the reflection symmetry (3.8) and hence they are double counted. To determine this number, we first must identify the number of configurations which are symmetric under the reflection. Symmetric rod structures can only occur for odd n and even n_+, with the middle centre corresponding to the horizon, in which case there are $\binom{(n-1)/2}{n_+/2}$ such symmetric configurations. Putting all this together, we find that the number of components of the moduli space of single black hole solutions with $L(p,1)$ topology, $\mathcal{M}_p^{n-1,1}$, is

$$N(\mathcal{M}_p^{n-1,1}) = \begin{cases} \frac{n}{2}\binom{n-1}{n_+} + \frac{1}{2}\binom{(n-1)/2}{n_+/2} & \text{if } n \text{ odd and } n_+ \text{ even} \\ \frac{n}{2}\binom{n-1}{n_+} & \text{otherwise} \end{cases}. \tag{3.14}$$

Summing over the possible p we find that the total number is

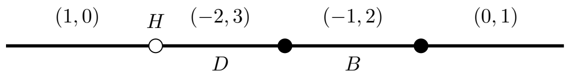
$$N(\mathcal{M}^{n-1,1}) = \sum_{n_+=0}^{n-1} N(\mathcal{M}_p^{n-1,1}) = \begin{cases} n2^{n-2} & \text{if } n \text{ is even} \\ n2^{n-2} + 2^{\frac{n-3}{2}} & \text{if } n \text{ is odd} \end{cases}. \tag{3.15}$$

Let us consider a few examples.

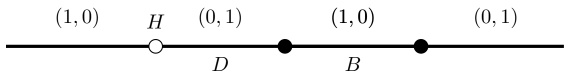
The simplest possibility is $n = 1$, which implies $n_+ = 0$ and $p = 1$ and hence the horizon topology is S^3. This of course corresponds to the BMPV black hole [15].

Now let us consider the $n = 2$ case. From (3.15) we find that there are 2 classes of two-centred single black hole solutions, whose rod structures are shown in Figure 3.2. The first of these, Figure 3.2(a) is the $L(2,1)$ black lens solution of [86]. Figure 3.2(b) corresponds to the known supersymmetric black ring solution [37].

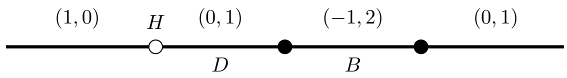
Next, for $n = 3$ we see there are seven distinct rod structures. These are depicted in



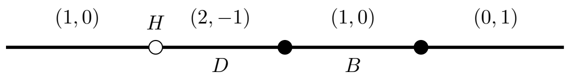
(a) $p = h_1 = 3$, $h_2 = -1$, $h_3 = -1$



(b) $p = h_1 = 1$, $h_2 = -1$, $h_3 = 1$

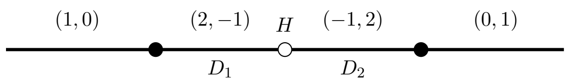


(c) $p = h_1 = 1$, $h_2 = 1$, $h_3 = -1$

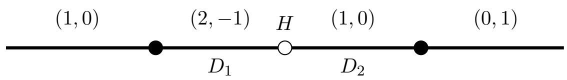


(d) $p = h_1 = -1$, $h_2 = 1$, $h_3 = 1$

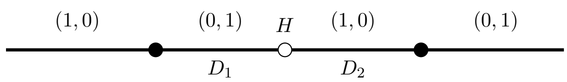
Figure 3.3: Rod structures for three-centred single black holes with the horizon at the first centre.



(a) $h_1 = -1$, $p = h_2 = 3$, $h_3 = -1$



(b) $h_1 = -1$, $p = h_2 = 1$, $h_3 = 1$



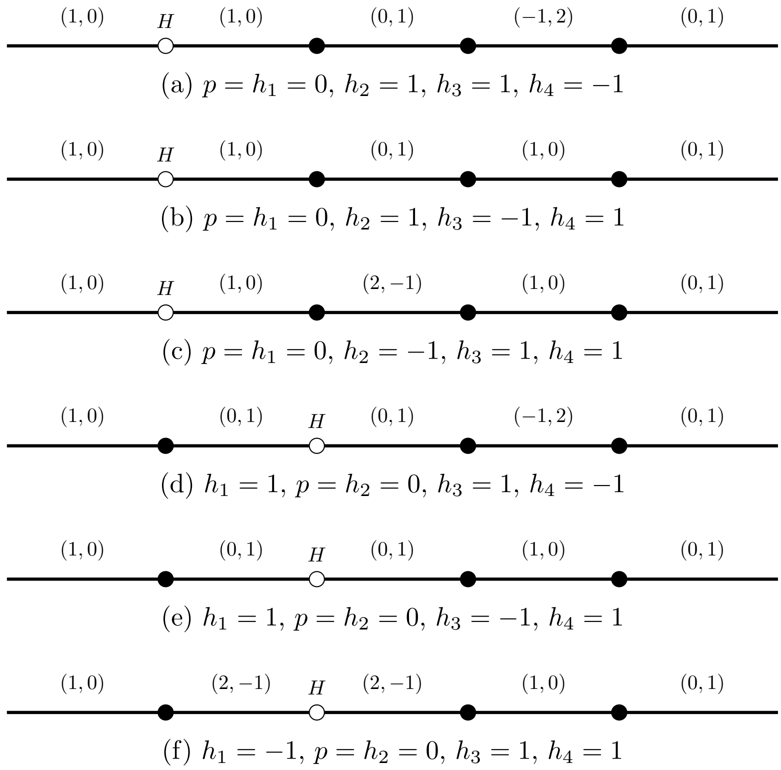
(c) $h_1 = 1$, $p = h_2 = -1$, $h_3 = 1$

Figure 3.4: Rod structures for three-centred single black holes with a central horizon.

Figures 3.3 and 3.4. There are two inequivalent black holes with a horizon of topology $L(3, 1)$, of which only Figure 3.3(a) corresponds to the solution constructed in [112]. There are five inequivalent S^3 black holes, of which only Figure 3.3(b) corresponds to the known S^3 black hole with bubble [85]. The other solutions had not previously been constructed.

More generally, we see that a single S^3 black hole, $p = \pm 1$, requires an odd number of centres. Such solutions correspond to a spherical black hole in a bubbling spacetime with $n-2$ bubbles (and one disc), or $n-3$ bubbles (and two discs), depending on which centre corresponds to the horizon, and have not been previously constructed.

Now consider single black hole solutions with $S^1 \times S^2$ horizon topology, so $p = 0$. These must have an even number of centres n and from the above we see that there are $\frac{n}{2}\binom{n-1}{n/2}$ inequivalent n-centred solutions with a single black ring. For even $n > 2$



(a) $p = h_1 = 0,\ h_2 = 1,\ h_3 = 1,\ h_4 = -1$

(b) $p = h_1 = 0,\ h_2 = 1,\ h_3 = -1,\ h_4 = 1$

(c) $p = h_1 = 0,\ h_2 = -1,\ h_3 = 1,\ h_4 = 1$

(d) $h_1 = 1,\ p = h_2 = 0,\ h_3 = 1,\ h_4 = -1$

(e) $h_1 = 1,\ p = h_2 = 0,\ h_3 = -1,\ h_4 = 1$

(f) $h_1 = -1,\ p = h_2 = 0,\ h_3 = 1,\ h_4 = 1$

Figure 3.5: Rod structures for four-centred single black ring solutions

we find there are an increasing number of inequivalent black ring solutions in bubbling spacetimes which have not previously been discussed. For example, in Figure 3.5, we list the six possible rod structures for four-centred single black ring solutions.

3.3.3 Multi black hole solutions

We will not consider the case of multi black holes in detail. Previously constructed examples in this class are the multi black rings [50], a double S^3 black hole [32] and more generally multi black lenses [113]. We emphasise that the multi extreme Reissner-Nordström and multi BMPV black hole solutions [22, 52] do not fit into our classification as they are not biaxisymmetric (they preserve at most $SO(3)$ rotational symmetry). Further these solutions violate the condition of smoothness on the horizon [21, 22].

3.4 Physical properties

Finally, let us turn to the physical properties of the solutions described in Theorem 2.5. The mass and angular momenta for the general solution can be obtained from the asymptotic expansion of the metric (see e.g. [44]) and are given by (using (2.152))

$$M_{\text{ADM}} = 3\pi\Big(\sum_{i=1}^{n}\ell_i + \frac{4}{9}m^2\Big)\,, \tag{3.16}$$

$$J_\psi = 2\pi\sum_{i=1}^{n}\Big(\frac{4}{9}m^2 k_i - m\ell_i + m_i\Big)\,, \tag{3.17}$$

$$J_\phi = 2\pi\sum_{i=1}^{n} z_i\Big(mh_i + \frac{3}{2}k_i\Big)\,. \tag{3.18}$$

The electric charge

$$Q = \frac{1}{4\pi}\int_{S^3}\star F = 2\sqrt{3}\pi\Big(\sum_{i=1}^{n}\ell_i + \frac{4}{9}m^2\Big) \tag{3.19}$$

satisfies the BPS bound $Q = \frac{2}{\sqrt{3}}M_{\text{ADM}}$.

As noted above, the general solution possesses nontrivial topology in the form of 2-cycles (bubbles, discs, or tubes) corresponding to the finite axis rods I_i, where $i = 1,\ldots,n-1$. The fluxes through these noncontractible 2-cycles C_i are given by

$$\Pi[C_i] = \frac{1}{4\pi}\int_{C_i} F = \frac{\sqrt{3}}{2}\frac{h_j m_j + \frac{1}{2}k_j\ell_j}{h_j\ell_j + k_j^2}\bigg|_{j=i}^{j=i+1}\,. \tag{3.20}$$

Note that for a corner z_j, the expression on the right hand side simplifies to $(h_j m_j + \frac{1}{2}k_j\ell_j)/(h_k\ell_j + k_j^2) = -k_j/h_j$. The nontrivial topology also allows us to define constant magnetic potentials Φ_i associated with each axis rod I_i by [87]

$$\iota_{v_i}F = \mathrm{d}\Phi_i\,, \tag{3.21}$$

where we fix $\Phi_i \to 0$ asymptotically. We find that, for $i = 1,\ldots,n-1$, the Φ_i evaluated on the corresponding axis rods are

$$q_i \equiv \Phi_i|_{I_i} = \frac{\sqrt{3}}{2}\Big((\chi|_{I_i} - 1)\sum_{j=1}^{i}k_j + (\chi|_{I_i} + 1)\sum_{j=i+1}^{n}k_j\Big)\,, \tag{3.22}$$

which are indeed constants.

Thus a solution with n centres carries the global charges Q, J_ψ, J_ϕ (with M_{ADM} fixed by Q) and also $n-1$ local magnetic potentials q_i (or magnetic fluxes $\Pi[C_i]$), leading to a total of $n+2$ physical charges. On the other hand, the dimension of the moduli space (3.7) for a solution with a single black hole is $n+1$ and for a soliton is $n-1$. Therefore, for a single black hole there must be a single constraint on the $n+2$ physical parameters, whereas for a soliton there must be three such constraints.

The constraints on the physical parameters can be seen more explicitly as follows.

Using the constraints on the parameters (2.154) one can show that

$$J_\phi = -2\pi \sum_{i=1}^{n} \sum_{j<i} \left(h_i m_j - m_i h_j - \tfrac{3}{2}(\ell_i k_j - k_i \ell_j)\right) . \tag{3.23}$$

The gauge freedom (3.2) implies that we can always set $k_j = 0$ for at least one $j \in \{1, \cdots, n\}$ (since at least one h_i must be nonvanishing due to (2.152)). Therefore, we may invert (3.22) to express the remaining $n-1$ parameters $k_{i\neq j}$ as linear combinations of the $n-1$ magnetic potentials q_i (one can check the matrix relating the two sets of quantities is indeed invertible). This gives a direct physical interpretation to the parameters k_i through the magnetic potentials. At every corner z_i the parameters ℓ_i, m_i are determined in terms of k_i and hence can also be expressed solely in terms of the magnetic potentials. In the case of a single black hole at position z_h, we can then invert (3.19) and (3.17) to express the parameters ℓ_h and m_h purely in terms of the physical parameters,

$$\ell_h = \ell_h(Q, q_i) , \qquad m_h = m_h(Q, J_\psi, q_i) . \tag{3.24}$$

Using these it is then clear that (3.23) implies the single constraint amongst the physical parameters of the form

$$J_\phi = J_\phi(Q, J_\psi, q_i) . \tag{3.25}$$

In the case of a soliton solution, all the parameters ℓ_i and m_i are completely determined by the k_i (and hence the q_i), which then implies the charge (3.19) and angular momenta (3.17, 3.23) can be expressed solely in terms of the magnetic potentials,

$$Q = Q(q_i) , \qquad J_\psi = J_\psi(q_i) , \qquad J_\phi = J_\phi(q_i) , \tag{3.26}$$

thus giving three constraints on the physical parameters as anticipated above.

Chapter 4

Supersymmetric black hole non-uniqueness in five dimensions

4.1 Introduction

The previous chapters have given a classification and detailed study of the moduli space of five-dimensional, supersymmetric solitons and black holes. The moduli space included a number of known solutions like the BMPV black hole [15], a supersymmetric black ring [37], a black lens [86], the bubbling soliton solutions [12], or a black hole with nontrivial topology (a nontrivial 2-cycle in the exterior spacetime) [85] which may be interpreted as a black hole sitting inside such a bubbling spacetime [79]. As was discussed in the preliminaries, some of these solutions have been of particular interest because they provide explicit examples of black hole non-uniqueness: Much like in the non-supersymmetric case, supersymmetric black rings in in five-dimensional $U(1)^3$-supergravity were found to exhibit non-uniqueness amongst themselves [38], providing the first example of supersymmetric black hole non-uniqueness (with a connected horizon; see [32, 50] for multi-black hole examples). The supersymmetric black ring solutions, however, always have unequal, nonzero angular momenta, and hence never carry the same charges as the BMPV solution. The more recently constructed black holes with nontrivial topology on the other hand provided an example of single black hole solutions that can in fact have the same asymptotic charges as the BMPV solution. This was studied in detail in [79]. In particular, not only were solutions found to allow for equal charges as the BMPV black hole, it was also found that in a region close to the BMPV upper spin bound the black holes with nontrivial topology in fact have entropy exceeding that of the BMPV black hole. This poses a challenge to the microscopic derivation of the BMPV entropy [15, 107]: The macroscopic entropy is recovered by counting microstates with the same charges as the black hole solution (see also section 1.2.1 of the preliminaries). It is thus unclear, why the counting reproduced the BMPV entropy, and not the entropy of one another solution with the same charges. A more detailed discussion of this problem is given in [79].

Equipped with the full moduli space laid out in the previous section, it is an interesting question to ask whether more such examples of black hole non-uniqueness—and in particular, more examples of solutions that may carry the same charges as the BMPV black hole—can be found. The objective of this chapter is to answer this question by performing a systematic analysis of the moduli space. Since in general the conditions on the parameters from Theorem 2.5 are nontrivial to deal with, we will focus on three-centred solutions. We shall find that these already provide a number of new examples of black hole non-uniqueness.

Our main results are as follows. We find that there are four 3-centred single black hole solutions that can have the same conserved charges as the BMPV black hole, one of which is the spherical black hole with nontrivial topology mentioned above [79, 85]. Furthermore, three of these may have an entropy greater than that of the BMPV solution near the BMPV upper spin bound, one of which is the previously known case [79]. The two new cases correspond to a distinct family of spherical black holes with nontrivial topology (with no bubble in the exterior, only disc topology surfaces ending on the horizon), and an $L(3, 1)$ black lens that is a distinct solution to the previously studied $L(p, 1)$ black lens [112]. In particular, our $L(3, 1)$ black lens provides the first example of a single black hole with nonspherical topology and the same conserved charges as the BMPV black hole.

This chapter is organised as follows. In section 4.2 we give details on three-centred solutions. In section 4.3 we analyse the moduli space of solutions which can have the same conserved charges as the BMPV black hole. Finally section 4.4 gives a comparison of the entropies of these black holes. We also give an Appendix with details on smoothness and causality of our solutions.

The contents of this chapter have first been presented in [17].

4.2 Three-centred solutions

As the main result of the first part of this thesis, we have shown that asymptotically flat, supersymmetric and biaxisymmetric black hole and soliton solutions to five-dimensional minimal supergravity must be of multi-centred Gibbons–Hawking type, see Theorem 2.5. Specifically, the harmonic functions are given by (2.151) and the remaining parameters in (2.151) must satisfy a number of constraint equations and inequalities arising from regularity of the spacetime (2.154–2.156). In general, these constraints are a complicated set of polynomial equations and inequalities, which makes studying solutions in more depth a difficult task (although see [5] for recent progress on the pure soliton case).

For single black hole solutions ($k = n - 1$, $l = 1$) with $n \leq 3$ the constraints are exactly solvable. The $n = 1$ solution reduces to the BMPV black hole; for $n = 2$ there are two possible solutions corresponding to the $L(2, 1)$ black lens [86] and the supersymmetric black ring [37], cf. section 3.3. Neither of these can have the same conserved charges as the BMPV black hole. We will here consider solutions with $n = 3$; in this case it is already known that there is at least one solution which may have the same charges as the BMPV black hole [79, 85].

For a single black hole solution with $n = 3$ the harmonic functions (2.151) are[1]

$$\begin{aligned} H &= \frac{h_0}{r} + \frac{h_1}{r_1} + \frac{h_2}{r_2}\,, & K &= \frac{k_1}{r_1} + \frac{k_2}{r_2}\,, \\ L &= 1 + \frac{\ell_0}{r} - \frac{h_1 k_1^2}{r_1} - \frac{h_2 k_2^2}{r_2}\,, & M &= -\frac{3}{2}(k_1 + k_2) + \frac{m_0}{r} + \frac{k_1^3}{2r_1} + \frac{k_2^3}{2r_2}\,, \end{aligned} \tag{4.1}$$

where we have written the $\mathbb{R}^3$ in spherical coordinates (r, θ, ϕ). We have fixed the origin $r = 0$ to be the position of the horizon and $r_1 = \sqrt{r^2 + a_1^2 - 2ra_1 \cos\theta} = 0$ and $r_2 = \sqrt{r^2 + a_2^2 - 2ra_2 \cos\theta} = 0$ are fixed points of the biaxial symmetry. Note that for notation to be in line with previous work on three-centred solutions, our index conventions are different to the ones used in the previous two chapters. As we will only consider single black hole solutions, we always choose the index $i = 0$ to specify the horizon, and $i = 1, 2$ to correspond to centres that are corners of the orbit space. In particular this means that indices do not necessarily reflect the order of the centres along the z-axis (as opposed to previous chapters). Now with these conventions, from asymptotic flatness and smoothness at the corners and the horizon we have $h_0 + h_1 + h_2 = 1$, $h_{1,2} = \pm 1$, $h_0 \in \mathbb{Z}$, and the remaining parameters are subject to the constraints

$$\frac{3}{2}(k_i - h_i(k_1 + k_2)) + \frac{-\frac{1}{2}h_0 k_i^3 + \frac{3}{2}k_i \ell_0 + h_i m_0}{|a_i|} + (-1)^i \frac{(h_2 k_1 - h_1 k_2)^3}{2|a_2 - a_1|} = 0\,, \quad i = 1, 2 \tag{4.2}$$

and inequalities

$$-h_0^2 m_0^2 + h_0 \ell_0^3 > 0\,, \tag{4.3}$$

$$h_i \left(1 + \frac{-h_0 k_i^2 + \ell_0}{|a_i|}\right) - \frac{h_1 h_2 (h_2 k_1 - h_1 k_2)^2}{|a_2 - a_1|} > 0\,, \qquad i = 1, 2\,. \tag{4.4}$$

which arise from the appropriate smoothness conditions at the three centres. Cross-sections of the horizon have area

$$A_H = 16\pi^2 \sqrt{\ell_0^3 h_0 - m_0^2 h_0^2}\,, \tag{4.5}$$

and the asymptotic charges of the solution are

$$Q = \frac{2}{\sqrt{3}} M_{\text{ADM}} = 2\sqrt{3}\pi \Big(-h_1 k_1^2 - h_2 k_2^2 + (k_1 + k_2)^2 + \ell_0 \Big)\,, \tag{4.6}$$

$$J_\psi = 2\pi \Big(\tfrac{1}{2}(k_1^3 + k_2^3) + (k_1 + k_2)^3 - \tfrac{3}{2}(k_1 + k_2)(h_1 k_1^2 + h_2 k_2^2 - \ell_0) + m_0 \Big)\,, \tag{4.7}$$

$$J_\phi = 3\pi \Big(a_1(k_1 - h_1(k_1 + k_2)) + a_2(k_2 - h_2(k_1 + k_2)) \Big)\,. \tag{4.8}$$

[1]We have exploited the gauge freedom (2.14) in the harmonic functions to set $k_0 = 0$.

Finally, for the solution to be smooth and stably causal, we require

$$K^2 + HL > 0\,, \qquad g^{tt} < 0 \tag{4.9}$$

in the domain of outer communication $r > 0$.

It is worth noting that, in general (4.2–4.4) are not sufficient to ensure positivity of the mass of the solution. Numerical checks suggest that $M_{\text{ADM}} > 0$ is indeed implied by (4.2–4.4) together with (4.9) (as must be the case from the BPS bound). Conversely, our checks also support the following conjecture: (4.2–4.4) together with $M_{\text{ADM}} > 0$ guarantee (4.9) are obeyed. If true, this would greatly simplify the study of the moduli space of supersymmetric black holes. In the Appendix, we present some numerical checks which support this conjecture. In any case, we will explicitly add

$$M_{\text{ADM}} = 3\pi\Big(- h_1 k_1^2 - h_2 k_2^2 + (k_1 + k_2)^2 + \ell_0 \Big) > 0 \tag{4.10}$$

to our set of constraints on the parameters and verify (4.9) numerically. In fact, if $h_0 = 3, h_1 = h_2 = -1$ the positive mass inequality (4.10) and smoothness condition $K^2 + HL > 0$ are automatically satisfied as a consequence of (4.2–4.4) (see Appendix for proof).

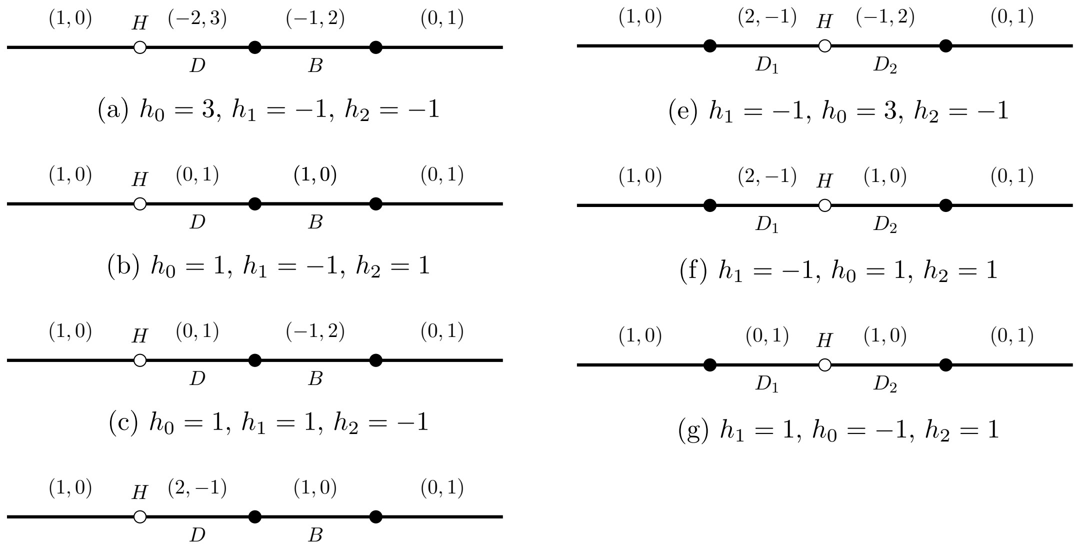
As mentioned above, we have fixed the position of the horizon to be the origin of the $\mathbb{R}^3$-base, $r = 0$. This can be done without loss of generality, however, we need to distinguish between the different possible orders of the centres along the z-axis: $0 < a_1 < a_2$, which corresponds to a horizon at the first centre, or $a_1 < 0 < a_2$, in which case the horizon is positioned between the other two centres. A potential horizon at the third centre is equivalent to one at the first centre, as they are related by the reflection symmetry (3.8) along the z-axis. As we have seen in section 3.3, with the above constraints on (h_0, h_1, h_2) one finds seven distinct rod structures, depicted in Figure 4.1. These lead to seven distinct regular black hole solutions.[2]

The rod structure determines the spacetime and horizon topology. If $0 < a_1 < a_2$ (Figure 4.1(a)–(d)), the axis rods $[0, a_1]$, $[a_1, a_2]$ correspond to a disc topology surface D ending on the horizon and an S^2-topology surface B (bubble), respectively. If $a_1 < 0 < a_2$ (Figure 4.1(e)–(g)) the axis rods both correspond to discs D_1 and D_2 ending on the horizon at $r = 0$. The dipole charges (3.22) for each axis rod I are

$$q_D = \sqrt{3} h_0 (k_1 + k_2)\,, \qquad q_B = \sqrt{3}(k_2 - h_2(k_1 + k_2)) \qquad \text{for } 0 < a_1 < a_2\,, \tag{4.11}$$

$$q_{D_1} = \sqrt{3}(-k_1 + h_1(k_1 + k_2))\,, \quad q_{D_2} = \sqrt{3}(k_2 - h_2(k_1 + k_2)) \qquad \text{for } a_1 < 0 < a_2\,. \tag{4.12}$$

[2]These are the same rod structures that have been depicted in the previous chapter in Figures 3.3, 3.4. To avoid confusion as we are working with a different convention for labelling the centres, they are repeated here.



(a) $h_0 = 3,\ h_1 = -1,\ h_2 = -1$

(b) $h_0 = 1,\ h_1 = -1,\ h_2 = 1$

(c) $h_0 = 1,\ h_1 = 1,\ h_2 = -1$

(d) $h_0 = -1,\ h_1 = 1,\ h_2 = 1$

(e) $h_1 = -1,\ h_0 = 3,\ h_2 = -1$

(f) $h_1 = -1,\ h_0 = 1,\ h_2 = 1$

(g) $h_1 = 1,\ h_0 = -1,\ h_2 = 1$

Figure 4.1: Rod diagrams for all seven three-centred single black hole solutions. (a)–(d) have the horizon at the first centre ($0 < a_1 < a_2$); (e)–(g) have a central horizon ($a_1 < 0 < a_2$). The numbers above each axis rod specify the biaxial Killing vector that vanishes on the rod with respect to the basis (v_-, v_+) where $v_\pm = \partial_\phi \mp \partial_\psi$.

The nontrivial 2-cycle structure is supported by the corresponding magnetic fluxes. Note that the dipole charges q_C are not conserved charges (they are not related to a symmetry of the solution).

Lastly, recall that the solutions defined by Figures 4.1(a) and 4.1(e) are black lenses with horizon topology $L(3,1)$, whereas Figures 4.1(b)–(d) and Figures 4.1(f)–(g) are spherical (S^3) black holes with nontrivial topology in the exterior. A $S^1 \times S^2$ horizon ($h_0 = 0$) cannot occur in this class of solutions (this is only possible for an even number of centres). Note that, as was pointed out in section 3.3, only two of these solutions have been previously studied: Figure 4.1(b) is the black hole with bubble solution in [79, 85], and Figure 4.1(a) is the black lens solution in [112].

We will also consider the soliton spacetimes in this class, that is, asymptotically flat everywhere smooth spacetimes with no black hole. These are constructed as above except the boundary condition at $r = 0$ is chosen to correspond to a corner of the orbit space rather than a horizon. The resulting constraints on the parameters are as above except (4.3) is now replaced by

$$\ell_0 = 0, \qquad m_0 = 0, \qquad h_0 \left(1 - \frac{h_1 k_1^2}{a_1} - \frac{h_2 k_2^2}{a_2}\right) > 0\,, \tag{4.13}$$

where now without loss of generality we may take $0 < a_1 < a_2$. The rod structures are more constrained in this case and there are only two inequivalent soliton solutions, see Figure 3.1 in section 3.3. For soliton solutions the 2-cycles C_1 and C_2 corresponding to the two axis rods $[0, a_1]$ and $[a_1, a_2]$ respectively are both topologically S^2.

4.3 Comparison with the BMPV black hole: Equal angular momenta

We now determine whether the conserved charges of the solutions described in the previous section can coincide with those of the BMPV black hole. The BMPV solution is a rotating black hole with equal angular momenta

$$|J_+| = |J_-| \,, \tag{4.14}$$

defined with respect to two orthogonal $U(1)^2$-Killing fields at infinity (normalised to 2π period), say $v_\pm = \partial_\phi \mp \partial_\psi$. Equality of J_+ and J_- is only defined up to a sign, as there is no fixed relative orientation of the corresponding orthogonal 2-planes. J_+ and J_- can be related to the angular momenta defined with respect to the Euler angles (ψ, ϕ) of the S^3 at infinity via $J_\pm = J_\phi \mp J_\psi$. Then (4.14) translates to

$$J_\phi = 0 \qquad \text{or} \qquad J_\psi = 0 \,. \tag{4.15}$$

An analysis of the constraints (4.2)–(4.4) and (4.10) on the parameter space for the three-centred solutions reveals that some of them allow for (4.15), whereas others always have strictly distinct charges to BMPV. The results of this analysis are summarised in Figure 4.2, which gives a list of all three-centred solutions, and whether or not (4.15) is allowed. We will discuss the four solutions for which (4.15) is possible in more detail below. For simplicity we will refer to the three solutions with an S^3-horizon that allow for (4.15) as "spherical black hole (with nontrivial topology) I, II, III", and to the $L(3,1)$ black hole that allows for (4.15) simply as "black lens". We emphasise though that the latter is a distinct solution to the previously studied $L(3,1)$ black hole depicted in Figure 4.1(a), for which (4.15) is never possible [112].

For each of the above solutions with equal angular momenta we have verified numerically that the smoothness and causality conditions (4.9) appear to be automatically satisfied as a consequence of (4.2)–(4.4) and (4.10). We give details of this in the Appendix. This lends support to our conjecture that (4.2)–(4.4) and (4.10) imply the smoothness and causality conditions.

To compare the solutions to the BMPV black hole, it is useful to define the dimensionless area and angular momentum

$$a_H \equiv \sqrt{\frac{3\sqrt{3}}{32\pi}\frac{A_H}{Q^{3/2}}} \,, \qquad \eta \equiv \sqrt{6\pi\sqrt{3}}\frac{J}{Q^{3/2}} \,, \tag{4.16}$$

where $J \equiv |J_+| = |J_-|$. For the BMPV black hole

$$0 \leq \eta_{\text{BMPV}} < 1 \,, \qquad a_{\text{BMPV}} = \sqrt{1-\eta^2} \,. \tag{4.17}$$

Therefore, a solution has the same conserved charges as the BMPV solution if and

		Horizon topology	Equal J?
$0 < a_1 < a_2$	$h_0 = 3,\ h_1 = h_2 = -1$	$L(3,1)$	$\times$
	$h_0 = 1,\ h_1 = -1,\ h_2 = 1$	S^3	$J_\phi = 0$ or $J_\psi = 0$
	$h_0 = 1,\ h_1 = 1,\ h_2 = -1$	S^3	$J_\psi = 0$
	$h_0 = -1,\ h_1 = h_2 = 1$	S^3	$\times$
$a_1 < 0 < a_2$	$h_1 = -1,\ h_0 = 3,\ h_2 = -1$	$L(3,1)$	$J_\phi = 0$
	$h_1 = -1,\ h_0 = 1,\ h_2 = 1$	S^3	$\times$
	$h_1 = 1,\ h_0 = -1,\ h_2 = 1$	S^3	$J_\phi = 0$

Figure 4.2: Three-centred single black hole solutions that allow for equal angular momenta.

only if (4.15) is satisfied and $0 \leq \eta < 1$. Our solutions also carry dipole charge associated to each 2-cycle C (bubble or disc), so it useful to also introduce a corresponding dimensionless dipole

$$\nu_C \equiv \sqrt{\frac{\pi}{\sqrt{3}}} \frac{|q_C|}{2Q^{1/2}} . \tag{4.18}$$

We are also interested in the soliton spacetimes which have equal angular momenta. In fact, the regularity constraints (4.2, 4.4, 4.10, 4.13) are compatible with the equal angular momentum condition (4.15) only for the soliton in Figure 3.1(b). For this case the constraints admit a unique 1-parameter family of solutions given by

$$k_2 = 0 , \qquad a_1 = \frac{k_1^2}{3} , \qquad a_2 = \frac{2k_1^2}{3}, \qquad k_1 \neq 0 . \tag{4.19}$$

This is the soliton previously studied in [79]. Its physical quantities simplify substantially,

$$Q = 4\sqrt{3}\pi k_1^2 , \qquad J_\psi = 6\pi k_1^3 , \qquad q_{C_1} = -q_{C_2} = -\sqrt{3}k_1 , \tag{4.20}$$

or in terms of the dimensionless quantities

$$\eta_s = \frac{3}{2\sqrt{2}} , \qquad \nu_s = \frac{1}{4} . \tag{4.21}$$

4.3.1 Spherical black hole with nontrivial topology I

Let us consider the first solution given in Figure 4.2 which admits equal angular momenta: the spherical black hole with $0 < a_1 < a_2$, $h_0 = 1$, $h_1 = -1$, $h_2 = 1$ (see Figure 4.1(b)). This is the previously studied spherical black hole with bubble [79, 85]. From (4.2) it follows that its physical charges satisfy the relation

$$J_\phi = -\frac{1}{2} Q q_D + \frac{\pi}{\sqrt{3}} q_B q_D (q_B - q_D) , \tag{4.22}$$

and we can express the area of the horizon (4.5) of the black hole in terms of physical quantities,

$$A_H = 8\pi^2 \left[\frac{1}{6\sqrt{3}\pi^3} \left(Q + \frac{4\pi}{\sqrt{3}} q_B q_D \right)^3 - \left(\frac{J_\psi + J_\phi}{\pi} - \frac{2}{\sqrt{3}} q_D q_B^2 \right)^2 \right]^{1/2} . \tag{4.23}$$

As can be seen from Figure 4.2, both J_ϕ and J_ψ can vanish in some subregion of the moduli space of the solution. We will study these two cases separately.

In general, we can always solve (4.2) for ℓ_0 and m_0, as $h_2 k_1 - h_1 k_2 = k_1 + k_2 \neq 0$ is guaranteed by the constraints on the parameters. The solution is then parameterised by the remaining 4 parameters (k_1, k_2, a_1, a_2). Setting either J_ϕ or J_ψ to zero imposes another constraint, resulting in a 3-parameter family of solutions in either case.

Case 1: $J_\phi = 0$

This case has been previously studied in [79], which we will briefly review here. As mentioned above, by solving (4.2) for ℓ_0 and m_0, one obtains a 4-parameter family of solutions, determined by the parameters (k_1, k_2, a_1, a_2). For the case at hand, we can readily solve $J_\phi = 0$ for k_2,

$$k_2 = -\frac{(2a_1 - a_2)k_1}{a_1} , \tag{4.24}$$

leaving a 3-parameter family of solutions specified by (a_1, a_2, k_1). One can express all physical quantities in terms of the dimensionless angular momentum η (4.16) and dipole $\nu \equiv \nu_B$ (4.18). In particular, the reduced area

$$a_H = \left[\left(16\nu^2 - 1\right)^3 - \left(\eta + 6\sqrt{2}\nu(8\nu^2 - 1) \right)^2 \right]^{1/2} . \tag{4.25}$$

The inequalities (4.3), (4.4) and (4.10) reduce to

$$\frac{1}{4} < \nu < \frac{1}{2\sqrt{2}} , \qquad \max\left(\eta_-(\nu), \frac{-1 + 40\nu^2 - 128\nu^4}{4\sqrt{2}\nu} \right) < \eta < \eta_+(\nu) , \tag{4.26}$$

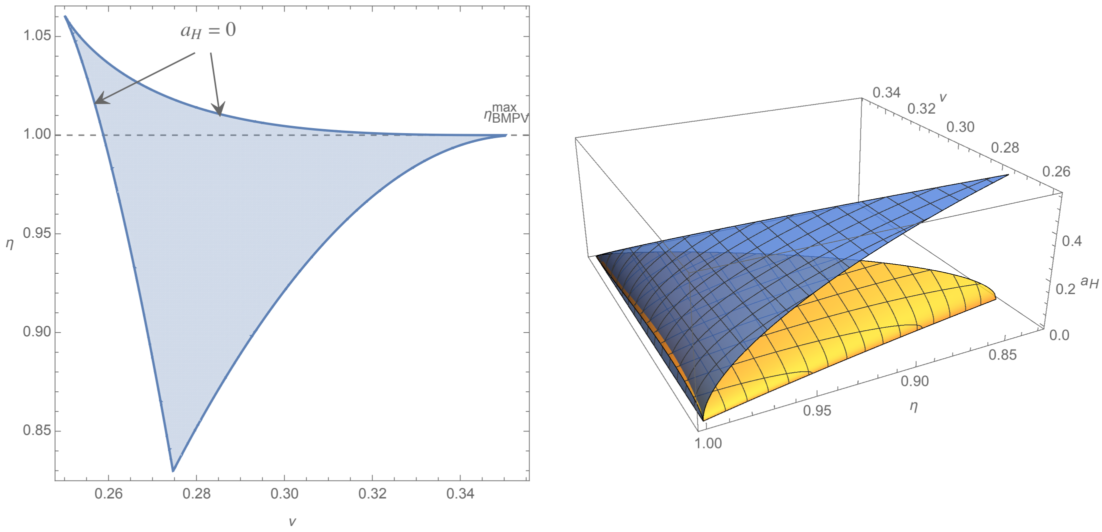
where

$$\eta_\pm(\nu) = \pm(16\nu^2 - 1)^{3/2} + 6\sqrt{2}\nu(1 - 8\nu^2) , \tag{4.27}$$

implying the range

$$\frac{-1 + 8\sqrt{2}}{8\sqrt{1 + \sqrt{2}}} < \eta < \frac{3}{2\sqrt{2}} . \tag{4.28}$$

The moduli space defined by (4.26) is the triangular region depicted in Figure 4.3(a). Notice that the area of the horizon vanishes along two of the boundary curves defined by $\eta_\pm(\nu)$ as shown in Figure 4.3(a). Moreover, the point $(\eta, \nu) = (\frac{3}{2\sqrt{2}}, \frac{1}{4})$ where those two curves intersect corresponds to the soliton solution (4.21). In fact, expanding the black hole solution near the soliton point reveals that one can interpret the solution in this limit as a small, nonrotating, extremal black hole sitting in the bubbling soliton



(a) (b)

Figure 4.3: (a) Moduli space for the $J_\phi = 0$ spherical black hole I ($0 < a_1 < a_2$, $h_0 = 1$, $h_1 = -1$, $h_2 = 1$). (b) Dimensionless area of the spherical black hole I (orange/lighter) and the BMPV black hole (blue/darker), in the region of overlap. Observe that $a_H > a_{\mathrm{BMPV}}$ in a narrow region close to $\eta = 1$.

geometry [79].

Finally, let us compare the solution with the BMPV black hole. In the region of overlap and near $\eta = 1$, i.e. for high angular momentum, there is a narrow region of the moduli space in which the black hole with nontrivial topology has an area (hence entropy) that is greater than that of the BMPV solution, see Figure 4.3(b).

Case 2: $J_\psi = 0$

This case has not been previously studied. As mentioned above, for this general family of solutions we can always solve the constraints for ℓ_0 and m_0. We now want to set $J_\psi = 0$. This, in general, imposes a more complicated constraint than vanishing of J_ϕ. Nevertheless, defining the dimensionless ratio $\alpha = a_1/a_2$, so $0 < \alpha < 1$, we can solve $J_\psi = 0$ (for a_2), resulting in a 3-parameter set of solutions specified by (α, k_1, k_2). The constraints on the parameters imply $k_1 \neq 0$ so it is convenient to introduce the dimensionless ratio $\kappa \equiv k_2/k_1$. In terms of these dimensionless variables we find that the constraints on the parameters can be reduced to [3]

$$-1 < \kappa < -0.773318\,, \qquad a_H^2 > 0\,, \qquad \frac{5+3\kappa}{6+6\kappa+3\kappa^2+\kappa^3} < \alpha < \alpha_+(\kappa)\,, \tag{4.29}$$

where

$$\alpha_+(\kappa) = \frac{-8-7\kappa+\kappa^3+(1+\kappa)\sqrt{64+152\kappa+149\kappa^2+70\kappa^3+13\kappa^4}}{2\kappa(2+\kappa)(3+3\kappa+\kappa^2)}\,, \tag{4.30}$$

[3] The upper limit for κ and lower limit for α are determined by the largest real root of $\kappa_0^3 + 9\kappa_0^2 + 18\kappa_0 + 9 = 0$ and $\alpha_0 = (5+3\kappa_0)/(6+6\kappa_0+3\kappa_0^2+\kappa_0^3)$.

the reduced area a_H is a complicated function of α, κ, and the resulting range of α is

$$0.995673 < \alpha < 1\,. \tag{4.31}$$

To translate this to physical parameters, let us again introduce a dimensionless dipole $\nu \equiv \nu_D$ (note that here we chose ν to be proportional to the dipole charge q_D rather than q_B). In terms of α and κ, the dimensionless charges η and ν are given by

$$\eta = \frac{3\sqrt{3(2+\kappa)}(1-\alpha)(1+\kappa)\left(-1+\alpha(2+\kappa)\right)\left(-5-3\kappa+\alpha(\kappa^3+3\kappa^2+6\kappa+6)\right)}{2\left[7+9\kappa+3\kappa^2-\alpha(8+7\kappa-\kappa^3)-\alpha^2\kappa(6+9\kappa+5\kappa^2+\kappa^3)\right]^{3/2}}\,, \tag{4.32}$$

$$\nu = \frac{\sqrt{3(2+\kappa)}(1-\alpha)(1+\kappa)}{2\sqrt{2}\left[7+9\kappa+3\kappa^2-\alpha(8+7\kappa-\kappa^3)-\alpha^2\kappa(6+9\kappa+5\kappa^2+\kappa^3)\right]^{1/2}}\,, \tag{4.33}$$

where positivity of each factor in the numerators and denominators is guaranteed by (4.29). In fact, within the region of interest (4.29), we can uniquely invert (4.32) and (4.33), giving

$$\kappa = -1 + 4\nu^2\left(-2\nu^2 + \sqrt{2\nu^2(1+2\nu^2) + \tfrac{\sqrt{2}}{3}\eta\nu}\right)^{-1} \tag{4.34}$$

and some complicated expression for α. To derive this inverse we used (4.22) to solve for q_B in terms of the other physical variables.

This also allows us to write the area in terms of η and ν as

$$\begin{aligned} a_H = \Bigg[&\left(1 + 8\nu^2 - 4\sqrt{2\nu^2(1+2\nu^2) + \tfrac{\sqrt{2}}{3}\eta\nu}\right)^3 \\ &- \left(\eta + 6\sqrt{2}\nu\left(1 + 4\nu^2 - 2\sqrt{2\nu^2(1+2\nu^2) + \tfrac{\sqrt{2}}{3}\eta\nu}\right)\right)^2\Bigg]^{1/2}\,, \end{aligned} \tag{4.35}$$

and the region (4.29) in terms of the physical parameters reduces to

$$\eta > 0\,, \qquad \nu > 0\,, \qquad a_H^2 > 0\,, \tag{4.36}$$

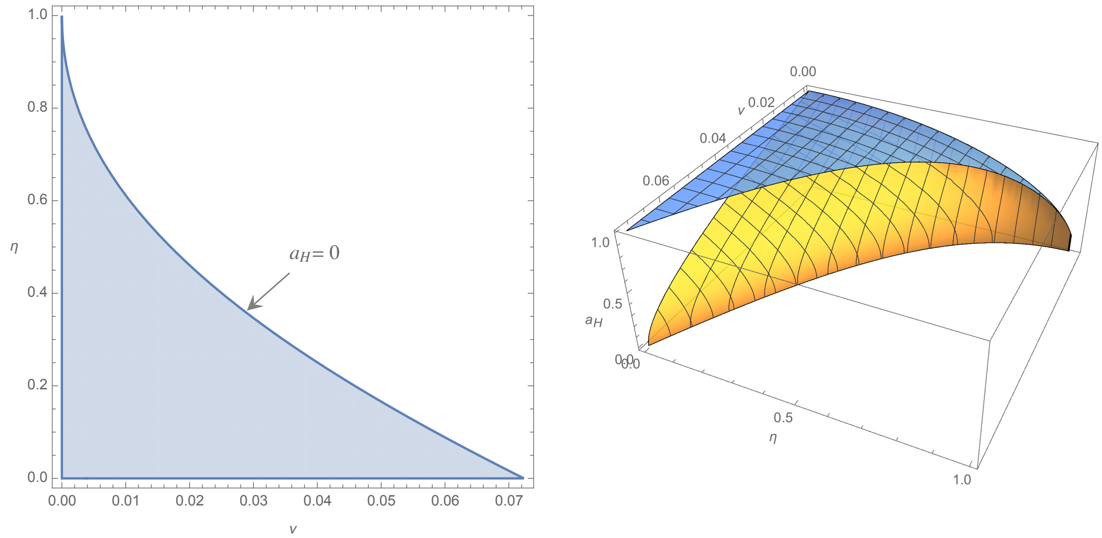
leading to the ranges

$$0 < \eta < 1\,, \qquad 0 < \nu < 0.072361\,. \tag{4.37}$$

The exact upper bound of ν is the unique positive root of $a_H(\eta = 0) = 0$. The resulting moduli space is the region depicted in Figure 4.4(a).

The upper bound for η at fixed ν (or vice-versa) corresponds to $a_H = 0$. The lower bound $\eta = 0$ corresponds to the curve

$$-5 - 3\kappa + \alpha\left(\kappa^3 + 3\kappa^2 + 6\kappa + 6\right) = 0. \tag{4.38}$$



(a) (b)

Figure 4.4: (a) Moduli space of the $J_\psi = 0$ spherical black hole I ($0 < a_1 < a_2$, $h_0 = 1$, $h_1 = -1$, $h_2 = 1$). (b) Dimensionless area (orange/lighter) as compared to that of the BMPV black hole (blue/darker) in the region of overlap. In this case $a_H < a_{\text{BMPV}}$ throughout the overlap region.

The lower bound $\nu = 0$ for $\eta > 0$ has no well defined meaning in terms of α and κ, as a consequence of the form of (4.32) and (4.33). The corner $(\eta, \nu) = (0, 0)$ corresponds to the limit $\alpha \to 1$, $\kappa \to -1$ (or equivalently in terms of the original parameters $a_1 \to a_2$, $k_1 \to -k_2$). This can be seen as follows.

Consider the limit $(\alpha, \kappa) \to (1, -1)$ along a curve within the region defined by (4.29). The limiting point $(\alpha, \kappa) = (1, -1)$ corresponds to the corner of the moduli space defined by the intersection of two boundary curves (4.38) and $a_H = 0$. We can expand each curve about this corner as a series in $(\kappa + 1)$: for (4.38) we find

$$\alpha = 1 - \frac{1}{2}(\kappa+1)^3 + \frac{3}{4}(\kappa+1)^4 - \frac{9}{8}(\kappa+1)^5 + \mathcal{O}((\kappa+1)^6)\,, \tag{4.39}$$

whereas for $a_H = 0$ we find

$$\alpha = 1 - \frac{1}{2}(\kappa+1)^3 + \frac{3}{4}(\kappa+1)^4 - \frac{3}{4}(\kappa+1)^5 + \mathcal{O}((\kappa+1)^6)\,. \tag{4.40}$$

Note that they agree up to fourth order in $(\kappa+1)$! Thus any smooth curve approaching the point $(1, -1)$ from within the moduli space will be of the form

$$\alpha = 1 - \frac{1}{2}(\kappa+1)^3 + \frac{3}{4}(\kappa+1)^4 + \alpha^{(5)}(\kappa+1)^5 + \mathcal{O}((\kappa+1)^6) \tag{4.41}$$

for some

$$-\frac{9}{8} < \alpha^{(5)} < -\frac{3}{4}\,. \tag{4.42}$$

Approaching the corner along such a curve, we find the physical charges

$$Q \to -\frac{4}{\sqrt{3}}(3+4\alpha^{(5)})k_1^2\pi\,, \qquad J_\phi \to 0\,, \qquad q_D \to 0\,, \qquad q_B \to -\sqrt{3}k_1\,, \tag{4.43}$$

and the area

$$A_H \to \frac{32\sqrt{2}}{3\sqrt{3}}\sqrt{-(3+4\alpha^{(5)})^3}|k_1|^3\pi^2\,. \tag{4.44}$$

Hence the dimensionless quantities tend to

$$a_H \to 1\,, \qquad \eta \to 0\,, \qquad \nu \to 0\,. \tag{4.45}$$

These are the physical quantities of a Reissner–Nordström solution. In terms of our solution this limit may be understood as arising from an effective "cancelling out" of two centres (recall $a_1 \to a_2, k_1 \to -k_2$ in this limit). We may thus interpret the solution near this limit as a Reissner–Nordström solution with a bubble added in the exterior of the black hole. This is in contrast to the solution with $J_\phi = 0$, which was interpreted as a black hole sitting in a soliton spacetime.[4]

For this class of solutions $0 < \eta < 1$, so the entire parameter space (4.36) overlaps with the BMPV solution. Furthermore, in contrast to the previous case, this shows that there are solutions with the same conserved charges as any rotating BMPV black hole (i.e. any $\eta > 0$), no matter how small η. Finally, one can show that

$$a_H < a_{\text{BMPV}} \tag{4.46}$$

throughout the moduli space (4.36). This is depicted in Figure 4.4(b). Thus for this class of solutions the entropy is always subdominant to the BMPV black hole.

4.3.2 Spherical black hole with nontrivial topology II

We will now study the second solution in Figure 4.2 which admits equal angular momentum: the spherical black hole with $0 < a_1 < a_2$, $h_0 = 1$, $h_1 = 1$, $h_2 = -1$ (see Figure 4.1(c)). This solution has not been studied before. The charges of the solution obey the constraint

$$J_\phi = -\frac{1}{2}Qq_D + \frac{\pi}{3\sqrt{3}}q_D(3q_B^2 - 3q_Bq_D + 2q_D^2)\,, \tag{4.47}$$

[4] It is of course still possible that this solution has a soliton limit in the more general moduli space of solutions with unequal angular momenta.

and the area of the horizon as a function of the charges is given by

$$A_H = 8\pi^2 \left[\frac{1}{6\sqrt{3}\pi^3} \left(Q + \frac{4\pi}{\sqrt{3}} q_D (q_D - q_B) \right)^3 - \left(\frac{J_\psi + J_\phi}{\pi} - \frac{2}{3\sqrt{3}} q_D (3q_B^2 - 6q_B q_D + 4q_D^2) \right)^2 \right]^{1/2} . \quad (4.48)$$

As shown in Figure 4.2, this solution admits equal angular momentum only if $J_\psi = 0$.

The analysis here is very similar to the $J_\psi = 0$ case of spherical black hole I. As in that case, we can always solve (4.2) for ℓ_0, m_0, since $h_2 k_1 - h_1 k_2 = -k_1 - k_2 \neq 0$ by the constraints on the parameters. Now, imposing $J_\psi = 0$, we obtain a three-parameter family of solutions parameterised by (α, k_1, k_2) where again $\alpha = a_1/a_2$ (hence by definition $0 < \alpha < 1$). As in the previous case we may introduce $\kappa \equiv k_2/k_1$ since the constraints on the parameters guarantee $k_1 \neq 0$. The resulting set of constraints on the parameters can be reduced to[5]

$$-1.307191 < \kappa < -1\,, \qquad a_H^2 > 0\,, \qquad \alpha_-(\kappa) < \alpha < \frac{1 + 5\kappa + 10\kappa^2 + 8\kappa^3}{2 + 8\kappa + 13\kappa^2 + 9\kappa^3}\,, \quad (4.49)$$

where

$$\alpha_-(\kappa) = \frac{1}{2\kappa^2(3 + 9\kappa + 7\kappa^2)} \Big[(1 + 2\kappa) \left(-2 - 5\kappa - 2\kappa^2 + 3\kappa^3 \right) - (1 + \kappa)\sqrt{(1 + 2\kappa)(2 + 3\kappa)(2 + 7\kappa + 10\kappa^2 + 7\kappa^3 + 6\kappa^4)} \Big]\,, \quad (4.50)$$

a_H is a complicated function of α, κ, and the resulting range of α is

$$0.995433 < \alpha < 1\,. \quad (4.51)$$

Defining $\nu \equiv \nu_D$ as in (4.18), the dimensionless angular momentum and dipole are given by

$$\eta = \frac{3}{2}\sqrt{3\kappa(1 + 2\kappa)}(\alpha - 1)(1 + \kappa) \times \frac{(1 + 2\kappa - \alpha\kappa) \left(1 + 5\kappa + 10\kappa^2 + 8\kappa^3 - \alpha \left(2 + 8\kappa + 13\kappa^2 + 9\kappa^3 \right) \right)}{\left[-1 - 5\kappa - 9\kappa^2 - 6\kappa^3 + \alpha\left(2 + 9\kappa + 12\kappa^2 + \kappa^3 - 6\kappa^4\right) + \alpha^2 \left(3 + 9\kappa + 7\kappa^2\right)\kappa^2 \right]^{3/2}}\,, \quad (4.52)$$

$$\nu = \frac{\frac{1}{2\sqrt{2}}\sqrt{3\kappa(1 + 2\kappa)}(\alpha - 1)(1 + \kappa)}{\left[-1 - 5\kappa - 9\kappa^2 - 6\kappa^3 + \alpha\left(2 + 9\kappa + 12\kappa^2 + \kappa^3 - 6\kappa^4\right) + \alpha^2 \left(3 + 9\kappa + 7\kappa^2\right)\kappa^2 \right]^{1/2}}\,, \quad (4.53)$$

[5]The exact lower limits for α and κ are given by the smallest real root of $19 + 150\kappa_0 + 537\kappa_0^2 + 1163\kappa_0^3 + 1590\kappa_0^4 + 1284\kappa_0^5 + 424\kappa_0^6 = 0$ and $\alpha_0 = (1 + 5\kappa_0 + 10\kappa_0^2 + 8\kappa_0^3)/(2 + 8\kappa_0 + 13\kappa_0^2 + 9\kappa_0^3)$.

where positivity of the numerators and denominators follows from the above inequalities. In the region of interest defined by (4.49), we may invert (4.52) and (4.53) to obtain

$$\kappa = \frac{-6\nu^2 + \sqrt{3}\sqrt{2\nu^2(3-10\nu^2) + \sqrt{2}\eta\nu}}{18\nu^2 - \sqrt{3}\sqrt{2\nu^2(3-10\nu^2) + \sqrt{2}\eta\nu}} \tag{4.54}$$

and (again) some complicated expression for α. To derive this inverse we used (4.47) to solve for q_B in terms of the other physical quantities.

This also allows us to write the dimensionless area of the horizon as

$$a_H = \Bigg[\left(1 + 8\nu^2 - \frac{4}{\sqrt{3}}\sqrt{2\nu^2(3-10\nu^2) + \sqrt{2}\eta\nu}\right)^3 \\ - \left(\eta + 2\sqrt{2}\nu \left(3 + 4\nu^2 - 2\sqrt{3}\sqrt{2\nu^2(3-10\nu^2) + \sqrt{2}\eta\nu}\right)\right)^2 \Bigg]^{1/2} , \tag{4.55}$$

and the moduli space (4.49) in terms of the physical variables is now given by

$$\eta > 0\,, \qquad \nu > 0\,, \qquad a_H^2 > 0\,. \tag{4.56}$$

This implies the ranges

$$0 < \eta < 1\,, \qquad 0 < \nu < 0.073674\,, \tag{4.57}$$

where the upper bound for ν is given by the positive root of $a_H(\eta = 0) = 0$. The resulting moduli space is the region depicted in Figure 4.5(a).

The upper bound for η at fixed ν corresponds to $a_H = 0$, whereas the lower bound for η corresponds to the curve

$$1 + 5\kappa + 10\kappa^2 + 8\kappa^3 - \alpha(2 + 8\kappa + 13\kappa^2 + 9\kappa^3) = 0\,. \tag{4.58}$$

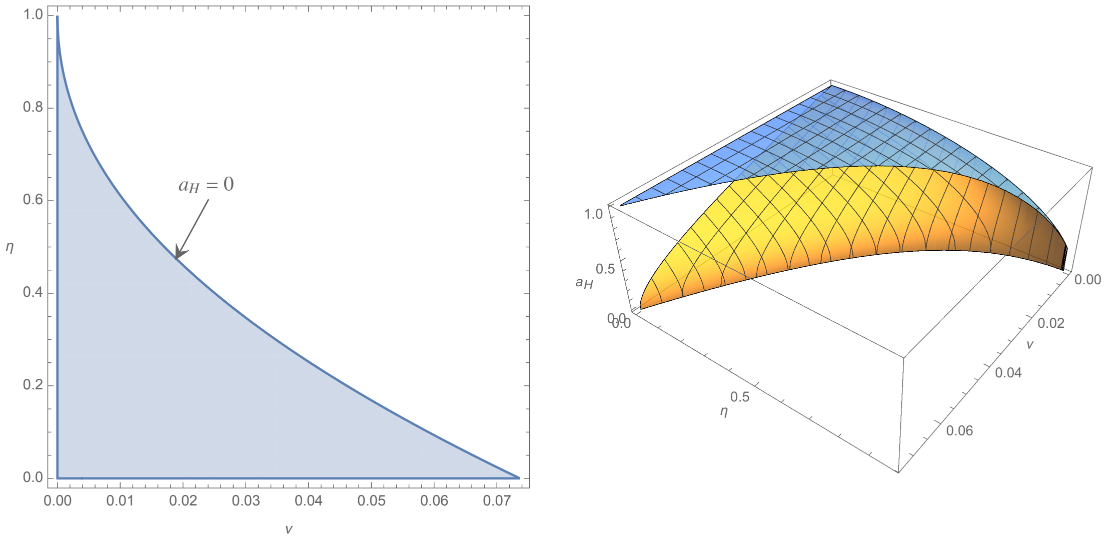
The corner $(\eta, \nu) = (0, 0)$ corresponds to the limit $\alpha \to 1$, $\kappa \to -1$ (or $a_1 \to a_2$, $k_1 \to -k_2$) where the two boundary curves (4.58) and $a_H = 0$ intersect. As in the previous case, we may expand any curve that approaches this limit from inside the moduli space to find

$$\alpha = 1 - \frac{1}{2}(\kappa+1)^3 + \frac{9}{4}(\kappa+1)^4 + \alpha^{(5)}(\kappa+1)^5 + \mathcal{O}((\kappa+1)^6) \tag{4.59}$$

for some

$$\frac{53}{8} < \alpha^{(5)} < 7\,, \tag{4.60}$$

where the lower bound corresponds to (4.58) and the upper bound to $a_H = 0$. We then



(a) (b)

Figure 4.5: (a) Moduli space for the $J_\psi = 0$ spherical black hole II ($0 < a_1 < a_2$, $h_0 = 1$, $h_1 = 1$, $h_2 = -1$). (b) Dimensionless area of the horizon (orange/lighter) as compared to that of the BMPV black hole (blue/darker) in the region of overlap. In this case $a_H < a_{\text{BMPV}}$ throughout the overlap region.

find the physical charges along such a curve are

$$Q \to \frac{16}{\sqrt{3}}(7-\alpha^{(5)})k_1^2\pi\,, \qquad J_\phi \to 0\,, \qquad q_D \to 0\,, \qquad q_B \to -\sqrt{3}k_1\,, \tag{4.61}$$

and the area

$$A_H \to \frac{256\sqrt{2}}{3\sqrt{3}}\sqrt{(7-\alpha^{(5)})^3}|k_1|^3\pi^2\,. \tag{4.62}$$

Therefore, in this limit

$$a_H \to 1\,, \qquad \eta \to 0\,, \qquad \nu \to 0\,, \tag{4.63}$$

just as we found in the previous case. Thus we also interpret this solution in this limit as a Reissner–Nordström black hole with a bubble added to the exterior.

Again, this family of solution has the same charges as any BMPV black hole with $\eta > 0$. Comparing the horizon area of the solution to that of the BMPV black hole again reveals that

$$a_H < a_{\text{BMPV}} \tag{4.64}$$

throughout the moduli space, see Figure 4.5(b), so this solution is also entropically subdominant to the BMPV black hole.

4.3.3 Black lens

We now study the third solution in Figure 4.2 which admits equal angular momenta: the $L(3,1)$ black lens defined by $a_1 < 0 < a_2$, $h_1 = -1$, $h_0 = 3$, $h_2 = -1$ (see

Figure 4.1(e)). This solution has not been previously studied. The physical charges obey the constraint

$$J_\phi = -\frac{1}{2}Q(q_{D_1}+q_{D_2}) + \frac{\pi}{3\sqrt{3}}\left(q_{D_1}^3+q_{D_2}^3+(q_{D_1}+q_{D_2})^3\right) , \tag{4.65}$$

and the area as a function of the charges is

$$\begin{aligned} A_H = 8\pi^2 \Bigg[& \frac{1}{2\sqrt{3}\pi^3}\left(Q - \frac{4\pi}{3\sqrt{3}}\left(q_{D_1}^2 + q_{D_1}q_{D_2} + q_{D_2}^2\right)\right)^3 \\ & - \left(\frac{3J_\psi}{\pi} + \frac{Q(q_{D_1}-q_{D_2})}{2\pi} + \frac{(q_{D_1}-q_{D_2})(2q_{D_1}+q_{D_2})(q_{D_1}+2q_{D_2})}{9\sqrt{3}}\right)^2\Bigg]^{1/2} . \end{aligned} \tag{4.66}$$

As shown in Figure 4.2 this solution admits equal angular momenta only if $J_\phi = 0$. We will now study this special case in detail.

Solving the constraints (4.2) together with the equal angular momentum condition $J_\phi = 0$ gives,

$$k_2 = k_1 , \qquad a_1 = -a_2 , \qquad m_0 = \frac{k_1}{2}(9a_2 - 3k_1^2 + 3\ell_0) . \tag{4.67}$$

The solution now depends on the remaining three parameters (a_2, k_1, ℓ_0). The inequalities (4.3), (4.4) then reduce to

$$3k_1^2 - \ell_0 - a_2 > 0 , \qquad 3\ell_0^3 - \frac{9}{4}k_1^2\left(9a_2 - 3k_1^2 + 3\ell_0\right)^2 > 0 , \tag{4.68}$$

which also guarantee (4.10) is obeyed.

The dimensionless angular momentum (4.16) and dipole $\nu \equiv \nu_{D_1}$ (4.18) are

$$\eta = \frac{9|k_1|(\ell_0 + 3k_1^2 + a_2)}{2(\ell_0 + 6k_1^2)^{3/2}}, \qquad \nu = \frac{3|k_1|}{2\sqrt{2}\sqrt{\ell_0 + 6k_1^2}} , \tag{4.69}$$

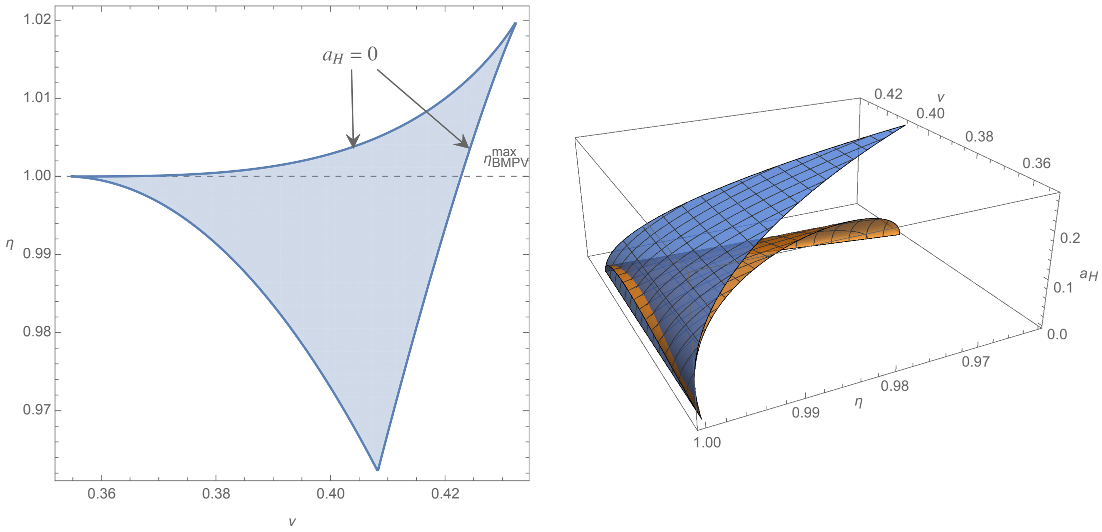
where positivity of both the numerators and denominators follows from the above inequalities (4.68). One can invert these to obtain

$$\frac{k_1^2}{a_2} = \frac{16\nu^3}{3(\sqrt{2}\eta - 6\nu + 16\nu^3)} , \qquad \frac{\ell_0}{a_2} = \frac{2\nu(3-16\nu^2)}{\sqrt{2}\eta - 6\nu + 16\nu^3} , \tag{4.70}$$

where the denominator is positive as a consequence of the inequalities. We may now express the moduli space defined by (4.68) in terms of the physical variables. We find this reduces to

$$\frac{1}{2\sqrt{2}} < \nu < \frac{\sqrt{3}}{4}, \qquad \max\left(\eta_-(\nu), \sqrt{2}(3\nu - 8\nu^3)\right) < \eta < \eta_+(\nu) , \tag{4.71}$$

$$\eta_\pm(\nu) = \pm\tfrac{1}{\sqrt{3}}\left(1 - \tfrac{16}{3}\nu^2\right)^{3/2} + \tfrac{4}{9\sqrt{2}}\nu\left(9 - 8\nu^2\right) , \tag{4.72}$$



(a) (b)

Figure 4.6: (a) Moduli space for the $L(3,1)$ black lens ($a_1 < 0 < a_2$, $h_1 = -1$, $h_0 = 3$, $h_2 = -1$), with $J_\phi = 0$. (b) Dimensionless area of the black lens (orange/lighter) and of the BMPV black hole (blue/darker), within the region of overlap. Observe that $a_H > a_{\mathrm{BMPV}}$ in a narrow region close to $\eta = 1$.

implying the range

$$\frac{5}{3\sqrt{3}} < \eta < \frac{5}{2\sqrt{6}} \,. \tag{4.73}$$

The resulting moduli space defined by this is the triangular region depicted in Figure 4.6(a). The reduced area is given by

$$a_H = \left[3\left(1 - \tfrac{16}{3}\nu^2\right)^3 - \left(3\eta - \tfrac{4}{3\sqrt{2}}\nu\left(9 - 8\nu^2\right)\right)^2\right]^{1/2} \,. \tag{4.74}$$

As in the case of the spherical black hole I with $J_\phi = 0$ discussed in section 4.3.1, the region is bounded by three curves, along two of which a_H vanishes (corresponding to the bounds $\eta_\pm(\nu)$). Similarly, the region extends beyond the BMPV upper bound $\eta = 1$. Furthermore, close to this bound we find a region in which the black lens has higher entropy than the BMPV black hole. The areas of the BMPV black hole and black lens are plotted in Figure 4.6(b).

It should be noted that in contrast to the spherical black hole I discussed in section 4.3.1, there is no possibility of a soliton limit of the black lens solution (the soliton requires $h_0 = \pm 1$).

4.3.4 Spherical black hole with nontrivial topology III

Let us now consider the fourth and final solution in Figure 4.2 with equal angular momenta: the spherical black hole with $a_1 < 0 < a_2$, $h_1 = 1$, $h_0 = -1$, $h_2 = 1$ (see Figure 4.1(g)). Again, this solution has not been previously analysed. The physical charges obey the constraint

$$J_\phi = -\frac{1}{2}Q(q_{D_1}+q_{D_2}) - \frac{\pi}{\sqrt{3}}q_{D_1}q_{D_2}(q_{D_1}+q_{D_2})\,, \tag{4.75}$$

and the area as a function of the charges is

$$A_H = \left[-\frac{1}{6\sqrt{3}\pi^3}\left(Q+\frac{4\pi}{\sqrt{3}}q_{D_1}q_{D_2}\right)^3 \right.$$
$$\left. - \left(\frac{J_\psi}{\pi} + \frac{Q(q_{D_2}-q_{D_1})}{2\pi} + \frac{q_{D_1}q_{D_2}(q_{D_2}-q_{D_1})}{\sqrt{3}}\right)^2\right]^{1/2}\,. \tag{4.76}$$

As shown in Figure 4.2 this solution admits equal angular momenta only if $J_\phi = 0$. The analysis of this solution very similar to that of the black lens solutions described in the previous section.

Solving the constraints (4.2) together with the equal angular momentum condition $J_\phi = 0$ gives

$$k_2 = k_1\,, \qquad a_1 = -a_2\,, \qquad m_0 = \frac{k_1}{2}(3a_2 - k_1^2 - 3\ell_0)\,, \tag{4.77}$$

so that the solution is again described by three parameters, (a_2, k_1, ℓ_0). The inequalities (4.3, 4.4, 4.10) constraining the parameter space then reduce to

$$k_1^2 + \ell_0 + a_2 > 0\,, \qquad -\ell_0^3 - \frac{1}{4}k_1^2\left(-3a_2 + k_1^2 + 3\ell_0\right)^2 > 0\,, \qquad 2k_1^2 + \ell_0 > 0\,. \tag{4.78}$$

The dimensionless angular momentum (4.16) and dipole $\nu \equiv \nu_{D_1}$ (4.18) are now

$$\eta = \frac{|k_1|(3\ell_0 + 5k_1^2 + 3a_2)}{2(\ell_0 + 2k_1^2)^{3/2}}, \qquad \nu = \frac{|k_1|}{2\sqrt{2}\sqrt{\ell_0 + 2k_1^2}}\,, \tag{4.79}$$

where positivity of the numerators follows from the above inequalities (4.78). Inverting these we obtain

$$\frac{k_1^2}{a_2} = \frac{48\nu^3}{(\sqrt{2}\eta - 6\nu + 16\nu^3)}\,, \qquad \frac{\ell_0}{a_2} = \frac{6\nu(1-16\nu^2)}{\sqrt{2}\eta - 6\nu + 16\nu^3}\,, \tag{4.80}$$

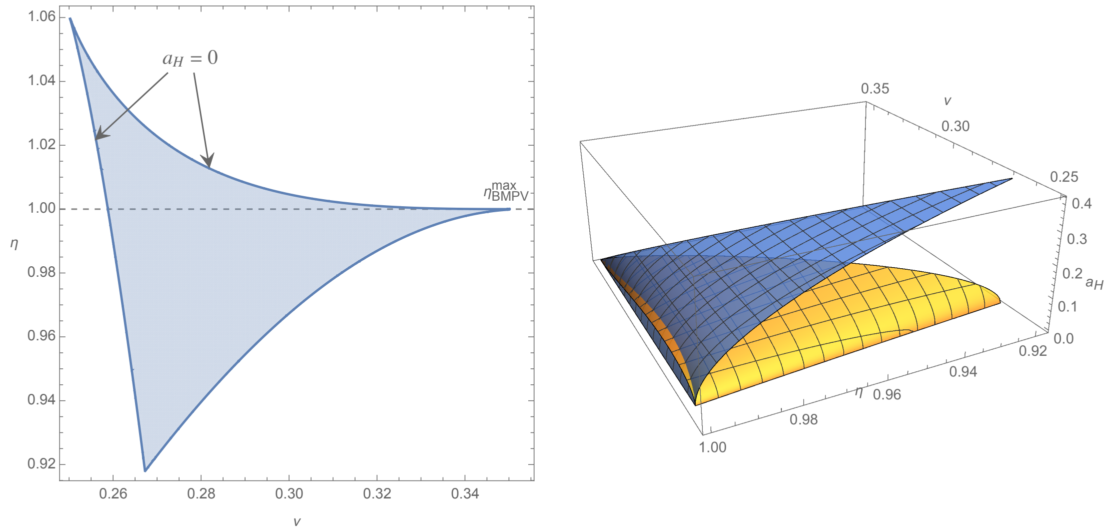
where the denominator is positive as a consequence of the inequalities.

We may now express the moduli space defined by (4.78) in terms of the physical variables. We find this reduces to

$$\frac{1}{4} < \nu < \frac{1}{2\sqrt{2}}, \qquad \max\left(\eta_-(\nu), \sqrt{2}(3\nu - 8\nu^3)\right) < \eta < \eta_+(\nu)\,, \tag{4.81}$$

where $\eta_\pm(\nu)$ are given again by (4.27), implying the range

$$\frac{17}{7\sqrt{7}} < \eta < \frac{3}{2\sqrt{2}}\,. \tag{4.82}$$



(a) (b)

Figure 4.7: (a) Moduli space for the $J_\phi = 0$ spherical black hole III ($a_1 < 0 < a_2$, $h_1 = 1$, $h_0 = -1$, $h_2 = 1$). (b) Dimensionless area of the spherical black hole III (orange/lighter) and the BMPV black hole (blue/darker) within the region of overlap. Again $a_H > a_{\text{BMPV}}$ in a narrow region close to $\eta = 1$.

The resulting moduli space is again a triangular region depicted in Figure 4.7(a). The reduced area is

$$a_H = \left[\left(16\nu^2 - 1\right)^3 - \left(\eta + 6\sqrt{2}\nu\left(8\nu^2 - 1\right)\right)^2\right]^{1/2} . \tag{4.83}$$

Note that the upper/lower bounds $\eta_\pm(\nu)$ again arise from positivity of the area $a_H^2 > 0$. The area of the black hole solution is plotted in Figure 4.7(b). It is clear that near $\eta = 1$ this solution also can have higher entropy than the BMPV black hole.

The moduli space in this case is very reminiscent of that of the spherical black hole I with $J_\phi = 0$ described in section 4.3.1. In fact, the expressions for the area a_H as a function of η and ν are identical for those two cases. However, the two moduli spaces (4.26) and (4.81) do not agree overall, as the remaining inequalities, which determine the other part of the lower boundary curve, are not equivalent for the two solutions. Furthermore it should be emphasised that the dipoles ν have a different meaning since the solutions have different spacetime topology (recall these are the magnetic potentials evaluated on a 2-cycle). It thus appears to be a curious coincidence that the area functions for these two solutions are the same in this special case. Indeed, inspecting the area as a function of the physical charges for these two solutions with $J_\phi \neq 0$, (4.23, 4.76), reveals that they are in fact distinct in general (modulo the constraint (4.22)).

Nevertheless, the similarity of the two solutions strongly suggests that the spherical black hole III will have a soliton limit. Indeed, the top left point on the boundary of the moduli space in Figure 4.7(a), which corresponds to the intersection of the upper and lower bounds arising from $a_H^2 > 0$, is again $(\eta, \nu) = (\frac{3}{2\sqrt{2}}, \frac{1}{4})$. This corresponds to the soliton solution (4.21) in a different gauge, namely, the polar coordinates are adapted to the middle centre instead of the first centre. The solution to the constraints, together with $J_\phi = 0$, now gives

$$k_2 = k_1 \,, \qquad a_2 = -a_1 = \frac{k_1^2}{3} \,, \tag{4.84}$$

and the charges are again given by (4.20) (where now C_1 and C_2 are the 2-cycles corresponding to the axis rods $[a_1, 0]$ and $[0, a_2]$ respectively).

We now consider this family of black hole solutions near the soliton point. The calculation is identical to that for the spherical black hole I [79]. In fact, since the moduli space near this point is determined by the same boundary curves, which correspond to the area vanishing, the expansion of the area of the black hole solution near the soliton point is the same as for the spherical black hole I, so we do not repeat it here. Thus just as for the spherical black hole I [79], we may interpret this as the area of a small nonrotating extremal black hole sitting in the soliton geometry.

4.4 Comparison of entropies

We have established in the previous section that there are three solutions which have the same conserved charges as the BMPV black hole and whose entropy near the BMPV bound, $\eta = 1$, may exceed that of the BMPV solution. Naturally, we want to compare the entropies of the different solutions in this region.

In particular, we are interested in determining the subregion of the moduli space with the same charges as BMPV, so $\eta < 1$, for which the area exceeds that of the BMPV back hole, i.e. $a_H \geq \sqrt{1-\eta^2}$. We found that the spherical black hole I and III happen to have the same area function $a_H(\eta, \nu)$, so in both cases this subregion is given by

$$\eta \geq \eta_{\text{crit}}^{\text{I, III}} \equiv \frac{1 + 20\nu^2 - 32\nu^4}{6\sqrt{2}\nu}, \qquad 0.259 \approx \frac{1}{2}\sqrt{2-\sqrt{3}} < \nu < \frac{1}{2\sqrt{2}} \,. \tag{4.85}$$

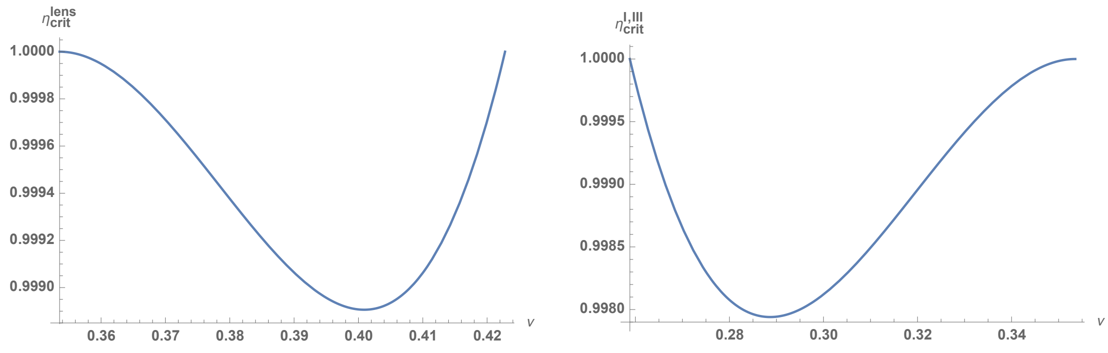
On the other hand, for the black lens we find it is given by [6]

$$\eta \geq \eta_{\text{crit}}^{\text{lens}} \equiv \frac{1}{2\sqrt{2}}\left(\nu(9-8\nu^2) + (1-8\nu^2)\sqrt{2-7\nu^2}\right), \qquad \frac{1}{2\sqrt{2}} < \nu < 0.423 \,. \tag{4.86}$$

It is worth noting that the curves η_{crit} are both very close to the BMPV upper bound $\eta = 1$, see Figure 4.8. To emphasise, all spherical black hole I and III and black lens solutions with parameters in the above respective ranges possess an entropy greater than the BMPV solution.

We will now compute the maximum entropy solutions for fixed η. Since we are interested in the region where the new solutions dominate over the BMPV solution,

[6] The exact upper limit of ν is given by the unique positive real root of $-9+72\nu_0^2-144\nu_0^4+128\nu_0^6 = 0$.



(a) (b)

Figure 4.8: $\eta_{\rm crit}$ for (a) the black lens and (b) the spherical black holes I and III.

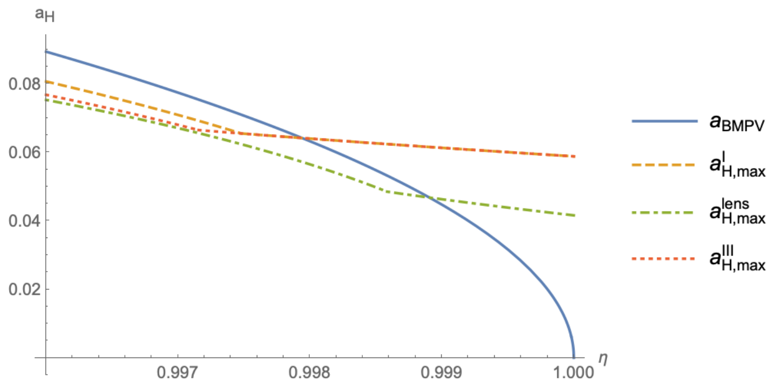


Figure 4.9: Maximum horizon areas of the different solutions near $\eta = 1$. Spherical black holes I and III have maximum areas exceeding those of the other solutions for $\eta > 0.997940$.

which occurs very near $\eta = 1$, it suffices to work in an expansion in $(1-\eta)$. To find the maximum we need the appropriate root of $\partial_\nu a_H = 0$ that gives a curve $\nu = \nu_*(\eta)$ along which a_H is maximised for fixed η. We find that near $\eta = 1$ these are given by

$$\nu_*^{\rm I,\,III}(\eta) \approx 0.284 + 2.025(1-\eta)\,, \qquad a_{H,\max}^{\rm I,\,III} \approx 0.059 + 2.404(1-\eta)\,, \tag{4.87}$$

$$\nu_*^{\rm lens}(\eta) \approx 0.406 - 3.604(1-\eta), \qquad a_{H,\max}^{\rm lens} \approx 0.042 + 4.364(1-\eta)\,. \tag{4.88}$$

The exact expressions for the maximum area are plotted in Figure 4.9. In particular, notice that near $\eta = 1$, the spherical black hole I and III have the same maximum area, which exceeds that of BMPV for

$$0.997940 \approx \frac{11\sqrt{2}}{9\sqrt{3}} < \eta < 1\,. \tag{4.89}$$

The black lens solution has a maximum area greater than that of BMPV for

$$0.998906 \approx \frac{37}{14\sqrt{7}} < \eta < 1\,, \tag{4.90}$$

however its maximum area is always less than that of the spherical black hole I and III solutions. Thus in the region (4.89) of the moduli space the spherical black hole I and III are both equally entropically favoured over the other solutions.

4.A Smoothness and causality in the domain of outer communication

As outlined in section 4.2, for a solution to be smooth and stably causal in the DOC, we require

$$K^2 + HL > 0\,, \qquad g^{tt} < 0\,. \tag{4.91}$$

The smoothness condition $K^2 + HL > 0$ is (away from the centres) equivalent to

$$\begin{aligned}\tilde{I} \equiv r^2 r_1 r_2 (K^2 + HL) = r_1 r_2 h_0 \ell_0 - r^2 h_1 h_2 (h_1 k_2 - h_2 k_1)^2 + r r_2 h_1 (\ell_0 - h_0 k_1^2) \\ + r r_1 h_2 (\ell_0 - h_0 k_2^2) + r r_1 r_2 h_0 + r^2 r_2 h_1 + r^2 r_1 h_2 > 0\,.\end{aligned} \tag{4.92}$$

The left hand side of this is in general a complicated function on the 2-dimensional orbit space $r > 0$, $0 \le \theta \le \pi$.

In fact, for the black lenses with $h_0 = 3$, $h_1 = h_2 = -1$, we can show that (4.92) is automatically satisfied as a consequence of (4.2)–(4.4) as follows. We can exploit the inequalities (4.4) to obtain

$$\tilde{I} \ge 3 r_1 r_2 \ell_0 + \left(-r + \frac{r_2 |a_1| + r_1 |a_2|}{|a_2 - a_1|} \right) r (k_1 - k_2)^2 + r \big(r_2 |a_1| + r_1 |a_2| + 3 r_1 r_2 - r r_2 - r r_1 \big)\,. \tag{4.93}$$

Since $h_0 \ell_0 = 3\ell_0 > 0$ is implied by positivity of the area of the horizon (4.3), the first term in (4.93) is always strictly positive. Furthermore, from basic triangle inequalities,

$$r_2|a_1| + r_1|a_2| + 3r_1 r_2 - r r_2 - r r_1 = r_1(r_2 - r + |a_2|) + r_2(r_1 - r + |a_1|) + r_1 r_2 \ge r_1 r_2 > 0\,, \tag{4.94}$$

showing that the third term is always positive. Lastly, one can show that

$$-r|a_2 - a_1| + r_2|a_1| + r_1|a_2| \ge 0\,, \tag{4.95}$$

so the second term is nonnegative. Therefore $\tilde{I} > 0$ everywhere away from the centres, so the smoothness of the spacetime is guaranteed without further restrictions.

For the other three-centred solutions it is not as straightforward to prove the smoothness condition. In general, establishing $K^2 + HL > 0$ requires input from the full condition for positivity of the horizon area, $h_0 \ell_0^3 - h_0^2 m_0^2 > 0$, the left hand side of which will be a complicated higher order polynomial in the remaining parameters once the constraint equations (4.2) are solved for ℓ_0 and m_0. Furthermore, in all cases we have been unable to prove the causality condition $g^{tt} < 0$. Therefore, for the remaining cases we have performed numerical checks that the smoothness and causality conditions (4.91) are satisfied as a consequence of (4.2)–(4.4) and (4.10), as follows.

Since we know $K^2 + HL > 0$ holds sufficiently far from the centres, we have checked the condition for 10^4 randomly chosen points within a region of radius $r_{\max} = 3\max(|a_1|, |a_2|)$ around the origin $r = 0$ (the position of the horizon). We have done this for a set of 10^4 randomly chosen parameters satisfying (4.2)–(4.4) and (4.10) for each of the seven general solutions listed in Figure 4.1 *and* the four special cases with equal angular momenta listed in Figure 4.2. The numerical checks confirm that in all cases the smoothness condition is satisfied without further restrictions on the parameters. In a similar manner (10^3 points for 10^4 parameters each) we have also checked that the causality condition $g^{tt} < 0$ holds without further restrictions on the moduli space.

This provides evidence for the following conjecture: (4.2)–(4.4) together with (4.10) imply (4.91) is automatically satisfied.

Chapter 5

Discussion

In this thesis we presented a complete classification of asymptotically flat, supersymmetric and biaxisymmetric solutions to five-dimensional minimal supergravity, which are regular on and outside an event horizon in the case of black hole solutions, or regular everywhere for spacetimes that do not contain a black hole. We have shown that such solutions must be of Gibbons–Hawking type with multi-centred harmonic functions. We have also found a refinement of the horizon topology theorem for this class of solutions, which restricts the topologies of cross sections of the horizon to one of S^3, $S^1 \times S^2$, or a lens space $L(p, 1)$. We have given a detailed analysis of the moduli space of solutions, recovering known solutions like the BMPV black hole, the supersymmetric black ring, the black lens and black hole with bubble solutions, as well as the known bubbling soliton solutions [11, 15, 37, 85, 86], but also constructing an infinite tower of new solutions. By a parameter count, we obtained the dimension of the moduli space for n-centred soliton or single black hole solutions. We have further given a detailed discussion of the geometry of our class of solutions, as well as its physical properties (charges). Solutions in general have a non-zero electric charge (related to the mass via the BPS relation), two angular momenta, and $n-1$ dipole charges related to nontrivial 2-cycles in the DOC.

While the conditions laid out in our classification theorem, Theorem 2.5, are necessary conditions for solutions to be physically well-behaved, to ensure smoothness and stable causality of our solutions one in general needs to impose additional conditions (2.16, 2.18) on the solution. For the solutions known prior to [16] it seemed that these smoothness and causality conditions were always automatically satisfied. A more detailed study of three-centred solutions showed, however, that this is not the case in general. In particular, the constraints (2.151–2.157) of Theorem 2.5 alone are not in general enough to guarantee positivity of the ADM mass (3.16). As must be the case, for the relevant solutions numerical evidence suggests that indeed (2.16, 2.18) imply $M_{\text{ADM}} > 0$. Even more so, it seems that in fact imposing positivity of the mass (in addition to the constraints of Theorem 2.5) is sufficient to guarantee smoothness and stable causality in the DOC. One can ponder whether this might be true for general,

n-centred solutions. Unfortunately however, we have so far not been able to prove this conjecture, even in the simpler, three-centred case. We certainly think that this is worth studying in more detail in the future. Despite this caveat, the classification presented in this thesis provides significant progress in solving the existence problem mentioned in the introduction for the case of supersymmetric solutions: For which rod structures do solutions exist? It is clear from our analysis that the allowed possibilities are heavily restricted. For example, for the case of a single black hole with one finite axis rod, the only allowed rod structures are the ones given in Figure 3.2, thereby excluding an infinite number of other potential rod structures. This in particular rules out any $L(p, 1)$ black hole with $p \neq 0, 2$ for single black holes with one finite axis rod.

Another possible route to follow in future work on the topic is a generalisation of Theorem 2.5 to a wider class of solutions. Specifically, a crucial assumption for our analysis was that solutions are biaxisymmetric, i.e. possess a $U(1)^2$-symmetry. This was a natural assumption, since all known single black hole solutions in five dimensions have this property. From the rigidity theorem, it is clear that solutions must at least have one $U(1)$ symmetry. There is so far no argument to rule out the existence of five-dimensional black holes with just one axial symmetry, however past attempts have typically failed to be smooth, see [80]. It would be interesting to investigate this further. Since our proof heavily relies on the fact that solutions can be written in terms of Weyl coordinates (this allows one to define a 2-dimensional orbit space and rod structures, concepts which govern both our near horizon as well as axes analysis), if solutions with less axial symmetry do exist it is to be expected that a full classification would need a rather different set of tools. The classification of near horizon geometries [100] used in section 2.4, however, remains valid also without the assumption of biaxisymmetry.

Another, and probably simpler, generalisation to envisage is that to $U(1)^3$-supergravity. This has been done for some solutions that fall into our classification (non-exhaustively see e.g. [38] for the black ring, [88] for the $L(2, 1)$ solution, or the recent paper [110] for the general $L(p, 1)$ case). It would be interesting to generalise our classification in the same way. In particular these generalised solutions allow for a *decoupling limit*, preserving the near-horizon geometry but introducing an AdS_3 factor in the asymptotic region which makes it possible to relate the solution to a two-dimensional CFT.

What remains far out of scope of the classification presented here are non-supersymmetric solutions. In general, a smaller variety of explicit solutions is known in the non-supersymmetric case. Specifically, so far there are no known, non-supersymmetric solutions with horizons of lens space topology, or solutions equivalent to the spherical black holes with nontrivial topology. A recent attempt to construct a non-supersymmetric vacuum black lens via inverse scattering failed to be regular due to the unavoidable existence of CTCs [111]. Despite these difficulties, from the uniqueness theorem [76], asymptotically flat, stationary and biaxisymmetric non-BPS solutions are specified by their asymptotic charges and rod-structure, much like the solutions discussed here, such that some of the arguments presented might carry over to the non-supersymmetric case.

In the latter part of this thesis, we analysed in detail the subset of three-centred solutions with a special focus on solutions whose charges can be equal to that of the BMPV black hole, thereby providing explicit examples of black hole non-uniqueness in higher dimensions. We found that this happens for four of the seven distinct three-centred solutions in the moduli space. One of these is the black hole with nontrivial spacetime topology studied previously in [79], however we have also found further, new examples with both spherical and lens space $L(3, 1)$ horizon topology. Further we have shown that two spherical black holes (both with nontrivial topology outside the event horizon) have equal entropy greater than that of the BMPV solution near the BMPV upper spin bound.

We have also found two examples of solutions that overlap with the BMPV black hole in the entire range of the BMPV spin parameter $0 \leq \eta_{\mathrm{BMPV}} < 1$, however which have entropy less that that of BMPV everywhere in that region. These solutions have a limit to the Reissner–Nordström solution for small angular momentum, such that in that region they may be interpreted as nonrotating black holes "dressed" with nontrivial topology in the exterior. For the two spherical solutions that only overlap with the BMPV black hole near the upper spin bound on the other hand, we found that they have a limit to a soliton solution, hence may be viewed as black holes sitting in a bubbling soliton spacetime.

The fact that there are multiple, single black hole solutions that have higher entropy than the BMPV black hole in some region of the phase space adds to the "single black hole entropy enigma" discussed in [79] (cf. [34] for the four-dimensional entropy enigma). In fact there might be infinitely many such solutions with higher entropy, as heuristic arguments in [79] suggest the entropy of solutions increases as further 2-cycles are added and there is no obvious upper bound on the number of centres. It remains unclear, why the BMPV entropy could correctly be reproduced by a counting of microstates of the same charges with no apparent further selection criteria on the string theory states. Clearly, to resolve these issues a better understanding of the relation between black hole solutions and BPS states in string theory (or in the CFT dual of the solutions' decoupling limit) is needed.

Bibliography

[1] B. P. Abbott et al. "Observation of Gravitational Waves from a Binary Black Hole Merger". *Phys. Rev. Lett.* 116.6 (2016), p. 061102. arXiv: 1602.03837 [gr-qc].

[2] K. Akiyama et al. "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole". *Astrophys. J.* 875.1 (2019), p. L1. arXiv: 1906.11238 [astro-ph.GA].

[3] J. Armas. "Uniqueness of Black Holes with Bubbles in Minimal Supergravity". *Class. Quant. Grav.* 32.4 (2015), p. 045001. arXiv: 1408.4567 [hep-th].

[4] D. Astefanesei and E. Radu. "Quasilocal formalism and black ring thermodynamics". *Phys. Rev.* D73 (2006), p. 044014. arXiv: hep-th/0509144 [hep-th].

[5] J. Avila, P. F. Ramirez, and A. Ruiperez. "One Thousand and One Bubbles". *JHEP* 01 (2018), p. 041. arXiv: 1709.03985 [hep-th].

[6] J. M. Bardeen, B. Carter, and S. W. Hawking. "The Four laws of black hole mechanics". *Commun. Math. Phys.* 31 (1973), pp. 161–170.

[7] K. Becker, M. Becker, and J. H. Schwarz. *String theory and M-theory: A modern introduction.* Cambridge University Press, 2006.

[8] J. D. Bekenstein. "Black holes and entropy". *Phys. Rev.* D7 (1973), pp. 2333–2346.

[9] J. D. Bekenstein. "Generalized second law of thermodynamics in black hole physics". *Phys. Rev.* D9 (1974), pp. 3292–3300.

[10] V. A. Belinsky and V. E. Sakharov. "Stationary Gravitational Solitons with Axial Symmetry". *Sov. Phys. JETP* 50 (1979), pp. 1–9.

[11] I. Bena and N. P. Warner. "Bubbling supertubes and foaming black holes". *Phys. Rev.* D74 (2006), p. 066001. arXiv: hep-th/0505166 [hep-th].

[12] I. Bena and N. P. Warner. "Black holes, black rings and their microstates". *Lect. Notes Phys.* 755 (2008), pp. 1–92. arXiv: hep-th/0701216 [hep-th].

[13] E. Bergshoeff, E. Sezgin, and P. K. Townsend. "Supermembranes and Eleven-Dimensional Supergravity". *Phys. Lett.* B189 (1987), pp. 75–78.

[14] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems". *Proc. Roy. Soc. Lond.* A269 (1962), pp. 21–52.

[15] J. C. Breckenridge, R. C. Myers, A. W. Peet, and C. Vafa. "D-branes and spinning black holes". *Phys. Lett.* B391 (1997), pp. 93–98. arXiv: hep-th/9602065 [hep-th].

[16] V. Breunhölder and J. Lucietti. "Moduli space of supersymmetric solitons and black holes in five dimensions". *Commun. Math. Phys.* 365.2 (2019), pp. 471–513. arXiv: 1712.07092 [hep-th].

[17] V. Breunhölder and J. Lucietti. "Supersymmetric black hole non-uniqueness in five dimensions". *JHEP* 03 (2019), p. 105. arXiv: 1812.07329 [hep-th].

[18] J. D. Brown and M. Henneaux. "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity". *Commun. Math. Phys.* 104 (1986), pp. 207–226.

[19] G. Bunting. "Proof of the Uniqueness Conjecture for Black Holes". PhD thesis. Univ. of New England, Armidale, N.S.W., 1983.

[20] A. C. Cadavid, A. Ceresole, R. D'Auria, and S. Ferrara. "Eleven-dimensional supergravity compactified on Calabi-Yau threefolds". *Phys. Lett.* B357 (1995), pp. 76–80. arXiv: hep-th/9506144 [hep-th].

[21] G. N. Candlish. "On the smoothness of the multi-BMPV black hole spacetime". *Class. Quant. Grav.* 27 (2010), p. 065005. arXiv: 0904.3885 [hep-th].

[22] G. N. Candlish and H. S. Reall. "On the smoothness of static multi-black hole solutions of higher-dimensional Einstein-Maxwell theory". *Class. Quant. Grav.* 24 (2007), pp. 6025–6040. arXiv: 0707.4420 [gr-qc].

[23] J. L. Cardy. "Operator Content of Two-Dimensional Conformally Invariant Theories". *Nucl. Phys.* B270 (1986), pp. 186–204.

[24] A. H. Chamseddine and W. A. Sabra. "Metrics admitting Killing spinors in five-dimensions". *Phys. Lett.* B426 (1998), pp. 36–42. arXiv: hep-th/9801161 [hep-th].

[25] P. T. Chrusciel. "On higher dimensional black holes with abelian isometry group". *J. Math. Phys.* 50 (2009), p. 052501. arXiv: 0812.3424 [gr-qc].

[26] P. T. Chrusciel, J. Lopes Costa, and M. Heusler. "Stationary Black Holes: Uniqueness and Beyond". *Living Rev. Rel.* 15 (2012), p. 7. arXiv: 1205.6112 [gr-qc].

[27] P. T. Chrusciel and L. Nguyen. "A uniqueness theorem for degenerate Kerr-Newman black holes". *Annales Henri Poincare* 11 (2010), pp. 585–609. arXiv: 1002.1737 [gr-qc].

[28] P. T. Chrusciel, H. S. Reall, and P. Tod. "On Israel-Wilson-Perjes black holes". *Class. Quant. Grav.* 23 (2006), pp. 2519–2540. arXiv: gr-qc/0512116 [gr-qc].

[29] K. Copsey and G. T. Horowitz. "The Role of dipole charges in black hole thermodynamics". *Phys. Rev.* D73 (2006), p. 024015. arXiv: hep-th/0505278 [hep-th].

[30] E. Cremmer. "Supergravities in 5 Dimensions". *In *Salam, A. (ed.), Sezgin, E. (ed.): Supergravities in diverse dimensions, vol. 1* 422-437. (In *Cambridge 1980, Proceedings, Superspace and supergravity* 267-282) and Paris Ec. Norm. Sup. - LPTENS 80-17 (80,rec.Sep.) 17 p. (see Book Index)*. 1980.

[31] E. Cremmer, B. Julia, and J. Scherk. "Supergravity Theory in Eleven-Dimensions". *Phys. Lett.* B76 (1978), pp. 409–412.

[32] P. M. Crichigno, F. Porri, and S. Vandoren. "Bound states of spinning black holes in five dimensions". *JHEP* 05 (2017), p. 101. arXiv: 1603.09729 [hep-th].

[33] J. R. David, G. Mandal, and S. R. Wadia. "Microscopic formulation of black holes in string theory". *Phys. Rept.* 369 (2002), pp. 549–686. arXiv: hep-th/0203048 [hep-th].

[34] F. Denef and G. W. Moore. "Split states, entropy enigmas, holes and halos". *JHEP* 11 (2011), p. 129. arXiv: hep-th/0702146 [hep-th].

[35] M. J. Duff and K. S. Stelle. "Multimembrane solutions of D = 11 supergravity". *Phys. Lett.* B253 (1991), pp. 113–118.

[36] H. Elvang. "A Charged rotating black ring". *Phys. Rev.* D68 (2003), p. 124016. arXiv: hep-th/0305247 [hep-th].

[37] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall. "A Supersymmetric black ring". *Phys. Rev. Lett.* 93 (2004), p. 211302. arXiv: hep-th/0407065 [hep-th].

[38] H. Elvang, R. Emparan, D. Mateos, and H. S. Reall. "Supersymmetric black rings and three-charge supertubes". *Phys. Rev.* D71 (2005), p. 024033. arXiv: hep-th/0408120 [hep-th].

[39] H. Elvang and P. Figueras. "Black Saturn". *JHEP* 05 (2007), p. 050. arXiv: hep-th/0701035 [hep-th].

[40] R. Emparan. "Rotating circular strings, and infinite nonuniqueness of black rings". *JHEP* 03 (2004), p. 064. arXiv: hep-th/0402149 [hep-th].

[41] R. Emparan and H. S. Reall. "A Rotating black ring solution in five-dimensions". *Phys. Rev. Lett.* 88 (2002), p. 101101. arXiv: hep-th/0110260 [hep-th].

[42] R. Emparan and H. S. Reall. "Generalized Weyl solutions". *Phys. Rev.* D65 (2002), p. 084025. arXiv: hep-th/0110258 [hep-th].

[43] R. Emparan and H. S. Reall. "Black Rings". *Class. Quant. Grav.* 23 (2006), R169. arXiv: hep-th/0608012 [hep-th].

[44] R. Emparan and H. S. Reall. "Black Holes in Higher Dimensions". *Living Rev. Rel.* 11 (2008), p. 6. arXiv: 0801.3471 [hep-th].

[45] P. Figueras and J. Lucietti. "On the uniqueness of extremal vacuum black holes". *Class. Quant. Grav.* 27 (2010), p. 095001. arXiv: 0906.5565 [hep-th].

[46] J. L. Friedman, K. Schleich, and D. M. Witt. "Topological censorship". *Phys. Rev. Lett.* 71 (1993). [Erratum: Phys. Rev. Lett.75,1872(1995)], pp. 1486–1489. arXiv: gr-qc/9305017 [gr-qc].

[47] G. J. Galloway and R. Schoen. "A Generalization of Hawking's black hole topology theorem to higher dimensions". *Commun. Math. Phys.* 266 (2006), pp. 571–576. arXiv: gr-qc/0509107 [gr-qc].

[48] J. P. Gauntlett. "Intersecting branes". *Dualities in gauge and string theories. Proceedings, APCTP Winter School, Sorak Mountain Resort, Seoul, Sokcho, Korea, February 17-28, 1997.* 1997, pp. 146–193. arXiv: hep-th/9705011 [hep-th].

[49] J. P. Gauntlett. "Branes, calibrations and supergravity". *Clay Math. Proc.* 3 (2004), pp. 79–126. arXiv: hep-th/0305074 [hep-th].

[50] J. P. Gauntlett and J. B. Gutowski. "Concentric black rings". *Phys. Rev.* D71 (2005), p. 025013. arXiv: hep-th/0408010 [hep-th].

[51] J. P. Gauntlett and J. B. Gutowski. "General concentric black rings". *Phys. Rev.* D71 (2005), p. 045002. arXiv: hep-th/0408122 [hep-th].

[52] J. P. Gauntlett, R. C. Myers, and P. K. Townsend. "Black holes of D = 5 supergravity". *Class. Quant. Grav.* 16 (1999), pp. 1–21. arXiv: hep-th/9810204 [hep-th].

[53] J. P. Gauntlett et al. "All supersymmetric solutions of minimal supergravity in five- dimensions". *Class. Quant. Grav.* 20 (2003), pp. 4587–4634. arXiv: hep-th/0209114 [hep-th].

[54] G. W. Gibbons and S. W. Hawking. "Gravitational Multi - Instantons". *Phys. Lett.* 78B (1978), p. 430.

[55] G. W. Gibbons and P. J. Ruback. "The Hidden Symmetries of Multicenter Metrics". *Commun. Math. Phys.* 115 (1988), p. 267.

[56] G. W. Gibbons and N. P. Warner. "Global structure of five-dimensional fuzzballs". *Class. Quant. Grav.* 31 (2014), p. 025016. arXiv: 1305.0957 [hep-th].

[57] G. W. Gibbons, D. Ida, and T. Shiromizu. "Uniqueness and nonuniqueness of static vacuum black holes in higher dimensions". *Prog. Theor. Phys. Suppl.* 148 (2003), pp. 284–290. arXiv: gr-qc/0203004 [gr-qc].

[58] U. Gran, J. Gutowski, and G. Papadopoulos. "Classification, geometry and applications of supersymmetric backgrounds". *Phys. Rept.* 794 (2019), pp. 1–87. arXiv: 1808.07879 [hep-th].

[59] R. Gueven. “Black p-brane solutions of D = 11 supergravity theory”. *Phys. Lett.* B276 (1992), pp. 49–55.

[60] M. Gunaydin, G. Sierra, and P. K. Townsend. “The Geometry of N=2 Maxwell-Einstein Supergravity and Jordan Algebras”. *Nucl. Phys.* B242 (1984), pp. 244–268.

[61] J. B. Gutowski and H. S. Reall. “General supersymmetric AdS(5) black holes”. *JHEP* 04 (2004), p. 048. arXiv: hep-th/0401129 [hep-th].

[62] D. Harlow. “Jerusalem Lectures on Black Holes and Quantum Information”. *Rev. Mod. Phys.* 88 (2016), p. 015002. arXiv: 1409.1231 [hep-th].

[63] T. Harmark. “Stationary and axisymmetric solutions of higher-dimensional general relativity”. *Phys. Rev.* D70 (2004), p. 124002. arXiv: hep-th/0408141 [hep-th].

[64] S. W. Hawking. “Black holes in general relativity”. *Commun. Math. Phys.* 25 (1972), pp. 152–166.

[65] S. W. Hawking. “Particle Creation by Black Holes”. *Commun. Math. Phys.* 43 (1975), pp. 199–220.

[66] S. W. Hawking. “Breakdown of Predictability in Gravitational Collapse”. *Phys. Rev.* D14 (1976), pp. 2460–2473.

[67] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2011.

[68] S. W. Hawking, M. J. Perry, and A. Strominger. “Soft Hair on Black Holes”. *Phys. Rev. Lett.* 116.23 (2016), p. 231301. arXiv: 1601.00921 [hep-th].

[69] M. Heusler et al. *Black Hole Uniqueness Theorems*. Cambridge Cultural Social Stud. Cambridge University Press, 1996.

[70] N. Hitchin. “Hyperkähler manifolds”. *Séminaire Bourbaki* 34 (1992), pp. 137–166.

[71] S. Hollands. “Black hole uniqueness theorems and new thermodynamic identities in eleven dimensional supergravity”. *Class. Quant. Grav.* 29 (2012), p. 205009. arXiv: 1204.3421 [gr-qc].

[72] S. Hollands, J. Holland, and A. Ishibashi. “Further restrictions on the topology of stationary black holes in five dimensions”. *Annales Henri Poincare* 12 (2011), pp. 279–301. arXiv: 1002.0490 [gr-qc].

[73] S. Hollands and A. Ishibashi. “Black hole uniqueness theorems in higher dimensional spacetimes”. *Class. Quant. Grav.* 29 (2012), p. 163001. arXiv: 1206.1164 [gr-qc].

[74] S. Hollands, A. Ishibashi, and R. M. Wald. "A Higher dimensional stationary rotating black hole must be axisymmetric". *Commun. Math. Phys.* 271 (2007), pp. 699–722. arXiv: gr-qc/0605106 [gr-qc].

[75] S. Hollands and S. Yazadjiev. "A Uniqueness theorem for 5-dimensional Einstein-Maxwell black holes". *Class. Quant. Grav.* 25 (2008), p. 095010. arXiv: 0711.1722 [gr-qc].

[76] S. Hollands and S. Yazadjiev. "Uniqueness theorem for 5-dimensional black holes with two axial Killing fields". *Commun. Math. Phys.* 283 (2008), pp. 749–768. arXiv: 0707.2775 [gr-qc].

[77] S. Hollands and S. Yazadjiev. "A Uniqueness theorem for stationary Kaluza-Klein black holes". *Commun. Math. Phys.* 302 (2011), pp. 631–674. arXiv: 0812.3036 [gr-qc].

[78] G. 't Hooft. "Dimensional reduction in quantum gravity". *Conf. Proc.* C930308 (1993), pp. 284–296. arXiv: gr-qc/9310026 [gr-qc].

[79] G. T. Horowitz, H. K. Kunduri, and J. Lucietti. "Comments on Black Holes in Bubbling Spacetimes". *JHEP* 06 (2017), p. 048. arXiv: 1704.04071 [hep-th].

[80] G. T. Horowitz and H. S. Reall. "How hairy can a black ring be?" *Class. Quant. Grav.* 22 (2005), pp. 1289–1302. arXiv: hep-th/0411268 [hep-th].

[81] V. E. Hubeny, M. Rangamani, and T. Takayanagi. "A Covariant holographic entanglement entropy proposal". *JHEP* 07 (2007), p. 062. arXiv: 0705.0016 [hep-th].

[82] S. Hwang. "A Rigidity Theorem for Ricci Flat Metrics". *Geometriae Dedicata* 71.1 (1998), pp. 5–17. URL: https://doi.org/10.1023/A:1005094911005.

[83] W. Israel. "Event horizons in static vacuum space-times". *Phys. Rev.* 164 (1967), pp. 1776–1779.

[84] W. Israel. "Event horizons in static electrovac space-times". *Commun. Math. Phys.* 8 (1968), pp. 245–260.

[85] H. K. Kunduri and J. Lucietti. "Black hole non-uniqueness via spacetime topology in five dimensions". *JHEP* 10 (2014), p. 082. arXiv: 1407.8002 [hep-th].

[86] H. K. Kunduri and J. Lucietti. "Supersymmetric Black Holes with Lens-Space Topology". *Phys. Rev. Lett.* 113.21 (2014), p. 211101. arXiv: 1408.6083 [hep-th].

[87] H. K. Kunduri and J. Lucietti. "The first law of soliton and black hole mechanics in five dimensions". *Class. Quant. Grav.* 31.3 (2014), p. 032001. arXiv: 1310.4810 [hep-th].

[88] H. K. Kunduri and J. Lucietti. "Black lenses in string theory" (2016). arXiv: 1605.01545 [hep-th].

[89] J. M. Maldacena. “Black holes in string theory”. PhD thesis. Princeton U., 1996. arXiv: hep-th/9607235 [hep-th]. URL: http://wwwlib.umi.com/dissertations/fullcit?p9627605.

[90] J. M. Maldacena. “The Large N limit of superconformal field theories and supergravity”. *Int. J. Theor. Phys.* 38 (1999). [Adv. Theor. Math. Phys.2,231(1998)], pp. 1113–1133. arXiv: hep-th/9711200 [hep-th].

[91] P. O. Mazur. “Proof of Uniqueness of the Kerr-Newman Black Hole Solution”. *J. Phys.* A15 (1982), pp. 3173–3180.

[92] P. O. Mazur. “Black Hole Uniqueness From a Hidden Symmetry of Einstein’s Gravity”. *Gen. Rel. Grav.* 16 (1984), p. 211.

[93] R. C. Myers. “Myers–Perry black holes”. *Black holes in higher dimensions.* Ed. by G. T. Horowitz. 2012, pp. 101–133. arXiv: 1111.1903 [gr-qc].

[94] R. C. Myers and M. J. Perry. “Black Holes in Higher Dimensional Space-Times”. *Annals Phys.* 172 (1986), p. 304.

[95] B. E. Niehoff and H. S. Reall. “Evanescent ergosurfaces and ambipolar hyperkähler metrics”. *JHEP* 04 (2016), p. 130. arXiv: 1601.01898 [hep-th].

[96] A. W. Peet. “TASI lectures on black holes in string theory”. *Strings, branes and gravity. Proceedings, Theoretical Advanced Study Institute, TASI’99, Boulder, USA, May 31-June 25, 1999.* 2000, pp. 353–433. arXiv: hep-th/0008241 [hep-th].

[97] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2007.

[98] J. Polchinski. “Dirichlet Branes and Ramond-Ramond charges”. *Phys. Rev. Lett.* 75 (1995), pp. 4724–4727. arXiv: hep-th/9510017 [hep-th].

[99] A. A. Pomeransky and R. A. Sen’kov. “Black ring with two angular momenta” (2006). arXiv: hep-th/0612005 [hep-th].

[100] H. S. Reall. “Higher dimensional black holes and supersymmetry”. *Phys. Rev.* D68 (2003). [Erratum: Phys. Rev.D70,089902(2004)], p. 024024. arXiv: hep-th/0211290 [hep-th].

[101] H. S. Reall. “Part 3 Black holes”. Lecture Notes. 2017.

[102] S. Ryu and T. Takayanagi. “Holographic derivation of entanglement entropy from AdS/CFT”. *Phys. Rev. Lett.* 96 (2006), p. 181602. arXiv: hep-th/0603001 [hep-th].

[103] R. Sachs. “Asymptotic symmetries in gravitational theory”. *Phys. Rev.* 128 (1962), pp. 2851–2864.

[104] R. K. Sachs. “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times”. *Proc. Roy. Soc. Lond.* A270 (1962), pp. 103–126.

[105] K. S. Stelle. “BPS branes in supergravity”. *High-energy physics and cosmology. Proceedings, Summer School, Trieste, Italy, June 2-July 4, 1997*. 1998. arXiv: hep-th/9803116 [hep-th].

[106] H. Stephani et al. *Exact solutions of Einstein's field equations*. Cambridge Monographs on Mathematical Physics. Cambridge: Cambridge Univ. Press, 2003. URL: http://www.cambridge.org/uk/catalogue/catalogue.asp?isbn=0521461367.

[107] A. Strominger and C. Vafa. “Microscopic origin of the Bekenstein-Hawking entropy”. *Phys. Lett.* B379 (1996), pp. 99–104. arXiv: hep-th/9601029 [hep-th].

[108] L. Susskind. “The World as a hologram”. *J. Math. Phys.* 36 (1995), pp. 6377–6396. arXiv: hep-th/9409089 [hep-th].

[109] F. R. Tangherlini. “Schwarzschild field in n dimensions and the dimensionality of space problem”. *Nuovo Cim.* 27 (1963), pp. 636–651.

[110] S. Tomizawa. “Multi-charged black lens” (2019). arXiv: 1905.07748 [hep-th].

[111] S. Tomizawa and T. Mishima. “A stationary and biaxisymmetric four-soliton solution in five dimensions” (2019). arXiv: 1902.10544 [hep-th].

[112] S. Tomizawa and M. Nozawa. “Supersymmetric black lenses in five dimensions” (2016). arXiv: 1606.06643 [hep-th].

[113] S. Tomizawa and T. Okuda. “Asymptotically flat multiblack lenses”. *Phys. Rev.* D95.6 (2017), p. 064021. arXiv: 1701.06402 [hep-th].

[114] S. Tomizawa, Y. Yasui, and A. Ishibashi. “Uniqueness theorem for charged rotating black holes in five-dimensional minimal supergravity”. *Phys. Rev.* D79 (2009), p. 124023. arXiv: 0901.4724 [hep-th].

[115] P. K. Townsend. “The eleven-dimensional supermembrane revisited”. *Phys. Lett.* B350 (1995), pp. 184–187. arXiv: hep-th/9501068 [hep-th].

[116] A. A. Tseytlin. “Extreme dyonic black holes in string theory”. *Mod. Phys. Lett.* A11 (1996), pp. 689–714. arXiv: hep-th/9601177 [hep-th].

[117] A. A. Tseytlin. “Harmonic superpositions of M-branes”. *Nucl. Phys.* B475 (1996), pp. 149–163. arXiv: hep-th/9604035 [hep-th].

[118] R. M. Wald. “The Thermodynamics of Black Holes”. *NATO Sci. Ser. II* 60 (2002), pp. 477–522.

[119] S. Weinberg. “Infrared photons and gravitons”. *Phys. Rev.* 140 (1965), B516–B524.